Transformation of Squeezed Quadratures under Feed-Forward Amplification and Post-Selection

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Abstract

Presented are calculations for quadrature transformation under feed-forward amplification and post-selection. A coherant state is interfered with a squeezed vacuum in a beamsplitter. Subsequently an emerging beam is amplitude-modulated by the amplitude of the second emerging beam. The modulated beam is then switched according to the result of a conditional measurement of the second beam. Wigner functions are used to visualise the beam states through the transformation.

1 Introduction

Squeezed light experiments have numerous applications to scientific research, in particular the field of quantum optics. Quantum optics is concerned with the circumstances in which light displays quantum, rather than classical, behaviour. There are many applications of quantum optics including quantum information, teleportation and fundamental tests of quantum mechanics.

In this report I define squeezed light and the Wigner function for beam states. A transformation is described in which the techniques of

- feed-forward quadrature amplification
- post-selection

are employed to modify a squeezed state. I then make calculations to predict the transformation statistics.

2 Squeezed Light

Two useful, physically measurable quantities associated with light are the amplitude and phase quadratures, X^+ and X^-. These quantities are analogous to the quantum mechanical operators $\hat{X^+}, \hat{X^-}$. It can be shown that a Heisenberg Uncertainty Principle (HUP) relates these operators such that

$$\Delta\hat{X^+}\Delta\hat{X^-} \geq 1. [1]$$

The case of equality represents the minimum uncertainties inherant in a simultaneous measurement of $\hat{X^+}, \hat{X^-}$. This is the *quantum noise limit* (QNL).

Squeezed light exists in a state where the uncertainty in one of the quadratures is less than the QNL. In order that the HUP is not violated, the other quadrature must have an increased uncertainty such that the HUP still holds. The next section describes a simple method of visualising squeezed states.

3 The Wigner Function and Simple Squeezed States

Wigner's function first appeared in his 1932 paper on thermodynamics.[2] Using our notation for quadratures it can be written

$$
\begin{aligned}
&W(X_1^+,\ldots,X_n^+,X_1^-,\ldots,X_n^-) = \\
&\quad\left(\frac{1}{2\pi}\right)^n \int\cdots\int_{-\infty}^{\infty} e^{\imath(X_1^- x_1+\ldots+X_n^- x_n)} \\
&\qquad \psi^*(X_1^+ + \frac{x_1}{2},\ldots,X_n^+ + \frac{x_n}{2}) \\
&\quad \psi(X_1^+ - \frac{x_1}{2},\ldots,X_n^+ - \frac{x_n}{2}) dx_1\ldots dx_n.
\end{aligned}
$$

The function describes the quadrature probability distribution of n modes of light, where $\psi(X_1^+,\ldots,X_n^+)$ is the amplitude wave function for the system. That is, given a set of values $X_1^+,\ldots,X_n^+,X_1^-,\ldots,X_n^-$, the Wigner function W is the probability of simultaneously measuring this set of values.

3.1 Vacuum State Wigner Function

Consider the single-mode vacuum state with HUP scaled to $\frac{1}{2}$.

$$
\begin{aligned}
\langle \hat{X}^+\rangle = \langle \hat{X}^-\rangle = 0, \\
\Delta\hat{X}^+ = \Delta\hat{X}^- = \frac{1}{\sqrt{2}}.
\end{aligned}
$$

The Wigner function is

$$
\begin{aligned}
W(X^+,X^-) = \left(\frac{1}{2\pi}\right) \int_{-\infty}^{\infty} e^{\imath x X^-} \\
\pi^{-\frac{1}{4}} e^{-\frac{1}{2}(X^+ + \frac{x}{2})^2} \pi^{-\frac{1}{4}} e^{-\frac{1}{2}(X^+ - \frac{x}{2})^2} dx.
\end{aligned}
$$

The integral evaluates to

$$
\frac{e^{-(X^+)^2-(X^-)^2}}{\pi}
$$

which is a 2-dimensional Gaussian and is plotted in Figure 1.

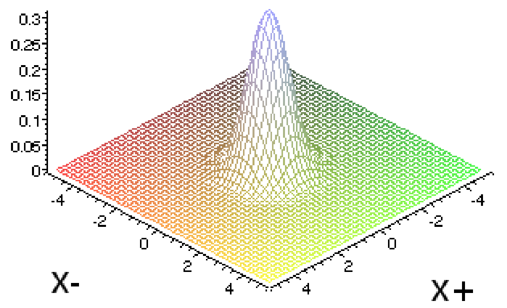

Figure 1: Vacuum State Wigner function

Plotted in Figure 2 is the Full Width at Half Maximum (FWHM) of this distribution.

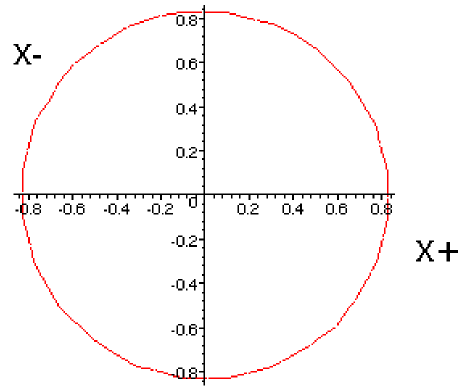

Figure 2: Vacuum State FWHM

The variances are proportional to the projection of the FWHM onto each axis. The plot shows that for a vacuum state the variance of each variable is identical, which was to be expected.

3.2 Coherant State Wigner Function

It can be shown that a coherant state is a vacuum state displaced in phase-space. That is,

$$
\begin{aligned}
&|\alpha\rangle = \hat{D}|0\rangle \\
&\therefore X_{coh}^+ = X_{vacuum}^+ - \Re(\alpha), X_{coh}^- = X_{vac}^- - \Im(\alpha) \\
&\therefore \psi_{coh}(X^+) = \psi_{vac}(X_{coh}^+) \\
&\therefore W_{coh}(X^+,X^-) = W_{vac}(X_{coh}^+,X_{coh}^-).
\end{aligned}
$$

Here α is the classical complex amplitude of the electric field. The Wigner function evaluates to

$$W_{coh}(X^+, X^-) = \frac{\mathrm{e}^{-\left(X^+ - \Re(\alpha)\right)^2 - \left(X^- - \Im(\alpha)\right)^2}}{\pi}.$$

Figure 3 plots the distribution.

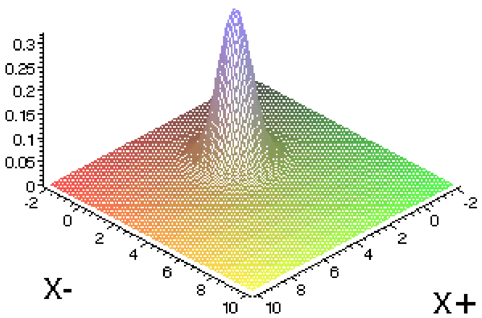

Figure 3: Coherant State Wigner Function

The centre of the distribution is displaced from the origin. This coherant state is quantum noise limited and therefore has an identical FWHM to the vacuum state.

3.3 Squeezed Vacuum Wigner Function

The squeezing operator $\hat{S} = \mathrm{e}^{\frac{1}{2}s\hat{\boldsymbol{a}}^2 - \frac{1}{2}s\hat{\boldsymbol{a}}^{\dagger 2}}$ acts to scale X^+, X^- as shown below:

$$\begin{aligned} &\hat{S}^\dagger \hat{X^+} \hat{S} \\ &= \hat{S}^\dagger(\hat{\boldsymbol{a}} + \hat{\boldsymbol{a}}^\dagger)\hat{S}^\dagger \\ &= \hat{\boldsymbol{a}}\cosh s - \hat{\boldsymbol{a}}^\dagger \sinh s + \hat{\boldsymbol{a}}^\dagger \cosh s - \hat{\boldsymbol{a}} \sinh s \\ &= \hat{\boldsymbol{a}}\mathrm{e}^{-s} + \hat{\boldsymbol{a}}^\dagger \mathrm{e}^{-s} \\ &= \mathrm{e}^{-s}\hat{X^+} \end{aligned}$$

where $\hat{\boldsymbol{a}}^\dagger, \hat{\boldsymbol{a}}$ are the creation and annihilation operators respectively and $s \in \mathbb{R}$ for purely amplitude or purely phase squeezing. Similarly

$$\hat{S}^\dagger \hat{X^-} \hat{S} = \mathrm{e}^{s}\hat{X^-}.$$

It can also be shown that

$$\hat{S}^\dagger \hat{X}^{\pm 2} \hat{S} = \mathrm{e}^{\mp 2s}\hat{X}^{\pm 2}.$$

The Wigner function of a squeezed vacuum is therefore

$$W_{sv}(X^+, X^-) = \frac{\mathrm{e}^{-\mathrm{e}^{-2s}X^{+2} - \mathrm{e}^{2s}X^{-2}}}{\pi}.$$

Figures 4,5 plot a squeezed vacuum with squeezing parameter $s = -0.7$. The condition $s < 0$ indicates amplitude squeezing.

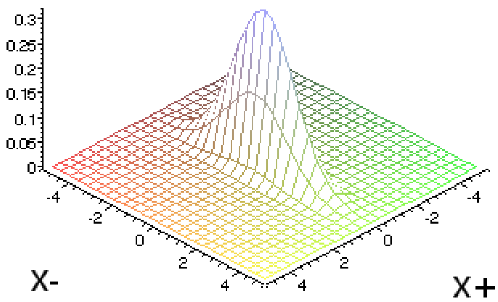

Figure 4: Squeezed Vacuum Wigner Function

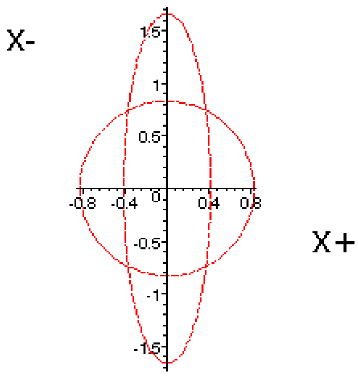

Figure 5: Squeezed Vacuum FWHM

Overlaid on Figure 5 is the FWHM of an unsqueezed vacuum for comparison. The squeezing in the amplitude quadrature relative to an unsqueezed vacuum is apparent, as is the conservation of the HUP by elongation of the phase quadrature.

4 Quadrature Transformations

An experiment is now considered where a coherant state is interfered with a squeezed vacuum in a beam splitter. The amplitude quadrature of one beam is measured, the signal amplified and used to modulate the second beam. This is feed-forward amplification (FFA). This measurement is also used to switch the modulated beam on and off. If a measurement returns a value deviating from the mean by less than a certain amount, the modulated beam is allowed through to detection. This is post-selection (PS).

4.1 Experimental Setup

Figure 6 describes the considered experimental setup. Sections 4.2, 4.3 and 4.4 calculate the transformation of the beam(s) at each point.

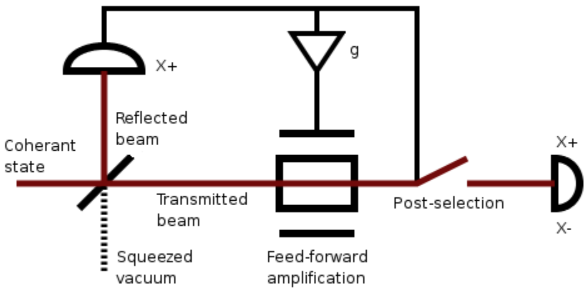

Figure 6: Experimental Setup

4.2 Interference

The effect of the beam splitter can be analysed by considering it as a rotational transform in phase space.[3] Using the conventions cited,

$$
\begin{aligned}
X_r^+ &= \eta X_{sv}^+ + \epsilon X_{coh}^+ \\
X_r^- &= \eta X_{sv}^- + \epsilon X_{coh}^- \\
X_t^+ &= \epsilon X_{sv}^+ - \eta X_{coh}^+ \\
X_t^- &= \epsilon X_{sv}^- - \eta X_{coh}^-
\end{aligned}
$$

where ϵ, η are the transmittivity and reflectivity such that $\epsilon^2 + \eta^2 = 1$.

The beams emerging from the beam splitter are potentially entangled. Writing the Wigner function of the system requires consideration of the beams' combined amplitude wave function. This combined wave function is the product of the squeezed vacuum and coherant wave functions, after transformation of the state variables by the beam splitter. The Wigner integral is therefore

$$
\begin{aligned}
&W(X_r^+, X_r^-, X_t^+, X_t^-) = \\
&\left(\frac{1}{4\pi^{\frac{5}{2}}}\right) \iint_{-\infty}^{\infty} e^{\imath e^s(\eta X_r^- + \epsilon X_t^-)x_1} \\
&e^{\imath(\epsilon X_r^- - \eta X_t^- - \Im(\alpha))x_2} \\
&e^{-\frac{1}{2}e^{-2s}(\eta X_r^+ + \epsilon X_t^+ - \frac{x_1}{2})^2} \\
&e^{-\frac{1}{2}(\epsilon X_r^+ - \eta X_t^+ - \frac{x_2}{2} - \Re(\alpha))^2} \\
&e^{-\frac{1}{2}e^{-2s}(\eta X_r^+ + \epsilon X_t^+ + \frac{x_1}{2})^2} \\
&e^{-\frac{1}{2}(\epsilon X_r^+ - \eta X_t^+ + \frac{x_2}{2} - \Re(\alpha))^2} dx_1 dx_2.
\end{aligned}
$$

Integrating results in a 4-dimensional Gaussian distribution with X_r^+, X_t^+ and X_r^-, X_t^- as pairs of covariant variables. This covariance, which is a quantitative measure of the beam entanglement (or correlation), can be visualised by plotting a 2-dimensional slice of the Wigner function (Figure 7). The slice is taken along the surface $X_r^- = \langle X_r^- \rangle, X_t^- = \langle X_t^- \rangle$.

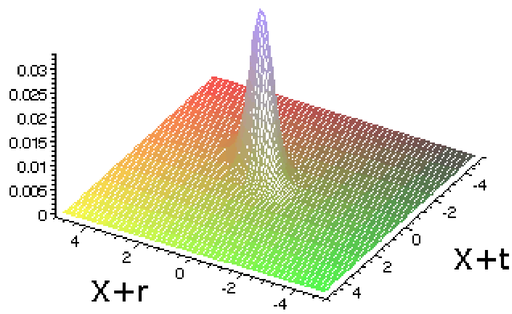

Figure 7: Amplitude Quadratures of Emerging Beams

The ellipse skew (covariance) can be seen even clearer in the plot of the FWHM, Figure 8. The vacuum noise is plotted for reference.

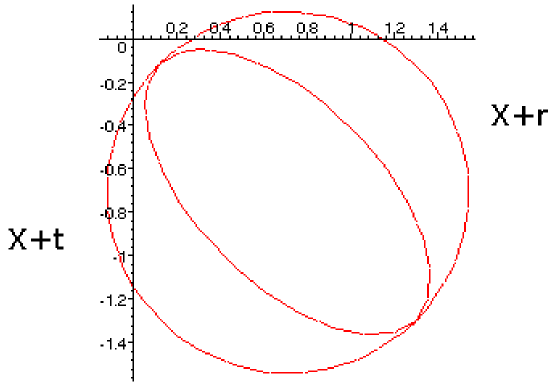

Figure 8: Amplitude Quadratures of Emerging Beams FWHM

The ellipse appearing inside the circle indicates that there exists amplitude states of the system that can be measured with greater precision than the QNL. This is to be expected since one of the inputs is amplitude-squeezed. A plot of the phase correlations (Figure 9) verifies non-violation of the HUP.

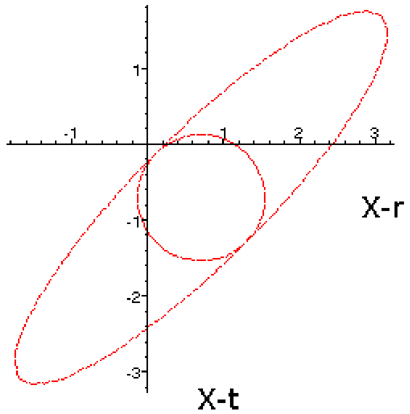

Figure 9: Phase Quadratures of Emerging Beams FWHM

4.3 Post-Selection

Post-selection allows the selection of only states measured less than a certain distance from the mean. The Wigner function after PS, $W_{ps}(X^+, X^-)$, is

$$\begin{cases} W(X_r^+, X_r^-, X_t^+, X_t^-) & \text{if } |\langle X_r^+\rangle - X_r^+| < a, \\ W(X_{vac}^+, X_{vac}^-) & \text{otherwise.} \end{cases}$$

Figures 10,11 describe the post-selected state. Here $a = 0.1$, approximately 15 times smaller than the vacuum FWHM.

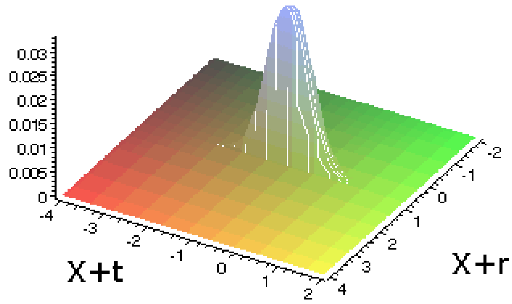

Figure 10: Post Selection Amplitude Quadratures X_r^+, X_{ps}^+

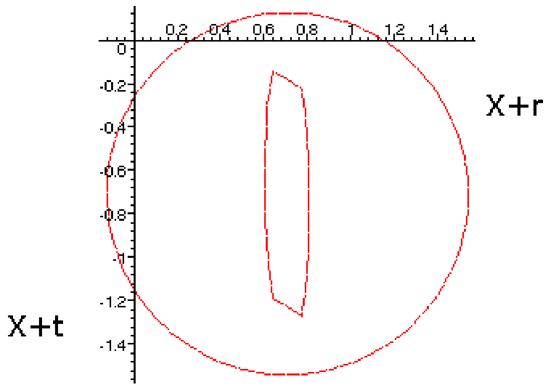

Figure 11: Post Selected Amplitude Quadratures FWHM

It can be seen that PS serves to increase the precision of quadrature measurement. The phase quadratures are unchanged apart from switching between X_t^+, X_t^- and X_{vac}^+, X_{vac}^- as the uncorrelated X_r^+ varies.

4.4 Feed-Forward Amplification

The effect of the amplifier and modulator in Figure 6 is to add a term gX_r^+ to X_t^-. The Wigner function of the system after modulation can therefore be written

$$W_{mod}(X_r^+, X_r^-, X_t^+, X_t^-) = W(X_r^+, X_r^-, X_t^+ + gX_r^+, X_t^-).$$

Figure 12 describes the effect of FFA for $g = 1.5$.

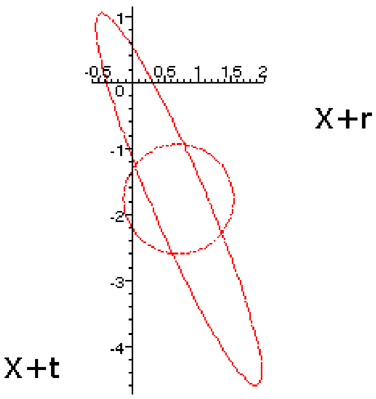

Figure 12: FFA Quadratures FWHM

FFA stretches the distribution outside of the QNL, but at the same time within the circle 'flattening' of the ellipse occurrs. The ellipse is also displaced by a distance $g\langle X_r^+\rangle$, since the signal consists of a DC current from the mean intensity plus variations corresponding to $g\partial X_r^+$.

4.5 FFA with PS

Figure 13 is a comparison of the vacuum noise, reflected beam, only-FFA and PS/FFA. The reflected beam's ellipse has been synthetically displaced by $g\langle X_r^+\rangle$ for ease of comparison.

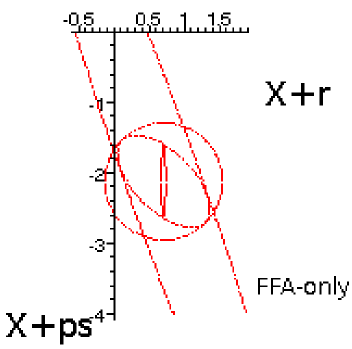

Figure 13: Comparison of Techniques ($a = 0.05, g = 2$)

The very narrow state is the PS/FFA beam. Observing the projection of this FWHM onto the X_{ps}^+ axis, it is clear that higher precision is available with PS/FFA than without.

5 Summary

Presented in this report is an explanation of squeezed light. The Wigner function is introduced and used to model and visualise squeezed states. The techniques of feed-forward amplification and post-selection are described. An experiment is proposed to observe the effects of FFA and PS on a squeezed state. Some of these effects are calculated using the theory developed. These calculations could be tested by conducting a squeezed light experiment as described.

6 Acknowledgments

I thank Nicolai Grosse for contributing to my understanding of the problem, and for useful discussions on Wigner functions and covariance. He will be testing predictions such as these experimentally.

References

[1] Bachor, H-A. and Ralph, T. C., *A Guide to Experiments in Quantum Optics*, Ed. 2, WILEY-VCH, Weinheim, **69** (2004).

[2] Wigner, E., *On the Quantum Correction for Thermodynamic Equilibrium*, Phys. Rev. **40**, 749 (1932).

[3] Leonhardt, Ulf, *Measuring the Quantum State of Light*, Cambridge University Press (1997).