Robust Quantum Control Theory

Stuart David Wilson

THE AUSTRALIAN
NATIONAL UNIVERSITY

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Declaration

This thesis is an account of research undertaken between January 2003 and November 2003 at The Department of Physics, Faculty of Science, of The Australian National University, Canberra, Australia.

Except where acknowledged in the customary manner, the material presented in this thesis is, to the best of my knowledge, original and has not been submitted in whole or part for a degree in any university.

Stuart David Wilson
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Acknowledgements

Being risk-insensitive to seeming passé, I'll follow tradition by thanking everyone who I believe has helped in some way to getting me here, being 2 hours 35 minutes before deadline.

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Abstract

The use of control theory in a LQG form for quantum mechanical situations has revealed a new conceptualization of knowledge, measurement and feedback. The benefits of state estimation by the Kalman filter has shown up in theoretical calculations for cooling and confining of an atom in a harmonic potential, surpassing previous proposals of direct feedback.

The goal of this thesis is to explore the possible application of a new form of filter/controller, the risk-sensitive LEQG filter, which has been shown to enjoy a sense of robustness against model uncertainties. An investigation is launched into some simple models, and a computer simulation has highlighted circumstances where the LEQG filter does provide better control.

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Chapter 1

Introduction

Ever since its rise out of the primordial soup, the hallmark of intelligence has been an interaction with its environment. It has long been the desire of humankind to manipulate their environment, whether it be agricultural, medicinal or aesthetically. To this extent, the concept of science, and its onslaught of technological advances, were inevitable. Over the years, from this progress and understanding has arisen the fields of quantum theory and control theory, which up until recently, have remained disjoint areas of science.

The theory

Control theory has been a classical conceptualization of feedback and control of a physical system, from the point of view of engineering. This theory requests the rigorous definition of many mathematically complex concepts; Itó-Stratonovich calculus, Ricatti equations. Even the innocuous term 'random variable' requires the construction of a probability space with a sense of measure. However, the resulting ideas from such formalism are powerful enough to justify this rigour; recursive, dynamic programming of an optimal control policy, the certainty-equivalence and separation principle, and the quintessential example of control policy, the Kalman filter. As a whole, control theory has had decades of refinement to provide a wealth of analytic tools to bring to bear on many problems.

The framework of quantum theory laid down earlier this century opened up a new description of the natural world, and introduced the idea of an inherent uncertainty in nature. Quite possibly this may not have been as dramatic a paradigm shift as another fundamental postulate, that of measurement. This requires the act of measurement carried out by an observer to *collapse* the wavefunction of the system onto an eigenstate of the observable. Some of the nouns in the previous sentence (observer, measurement) have no precise physical definition, while others (wavefunction, eigenstate, observable) are only abstract mathematical concepts typical of quantum theory. These two inherently quantum ideas, irreducible uncertainty and measurement intervention, seem almost contradictory to the ideals of control theory.

How can one control a system when the mere acts of observation affects the system? In recent decades much theoretical work has been performed that allows the merger of these fields and a resolution of the paradox of measurement.

This thesis will present results of a simulation of a risk-sensitive filter as a control system for a quantum system. In Ch.II the ideas of measurement theory will be introduced resulting in a stochastic master equation, an unusual entity that exemplifies quantum measurement. Ch.III covers the basic elements of the quantum systems under measurement. Ch.IV illuminates the ideas of control theory, and these are applied to the quantum domain in Ch.V, covering all of the current ideas of quantum control theory.

In Ch.VI the concept of a risk-sensitive filter with *robust* properties is discussed, and Ch.VII presents the results of a computer simulation designed to show an improvement in performance of the risk-sensitive filter over the risk-neutral Kalman filter. Ch.VIII concludes the thesis.

Chapter 2

Measurement Models

The process of measurement implies an interaction with the system under observation. By being able to model the effects of measurement as well as make some form of prediction of the results, the theory of continuous position measurement is as mathematically elegant as it is philosophically challenging.

This chapter covers some of the early ideas of position measurement, and the modern formalism of measurement, often referred to as 'operations and effects'. The end result is a stochastic master equation for bosonic operators which are coupled to the system under observation.

The next chapter will spell out how these operators relate to the position of the system under observation, and how homodyne detection will reveal this information.

2.1 Introduction

Some of the original theoretical difficulties in describing position measurement lay in the formal rules of Quantum Mechanics. For example, if one makes a measurement of position of an atom, then the wavefunction describing that atom must collapse onto one of the eigenstates of the position operator. Yet formally there are no normalized position eigenstates. Perhaps one could consider an infinite basis over the variable x, such as

$$|\psi(x)\rangle = \int dx' \, \delta(x - x')|x'\rangle$$

However, in momentum space this becomes unnormalizable

$$\begin{aligned} |\phi(k)\rangle &= \frac{1}{\sqrt{2\pi}} \int dx' \, \delta(x - x') \int dk \, e^{-ikx'}|k\rangle \\ &= \frac{1}{\sqrt{2\pi}} \int dk \, e^{-ikx}|k\rangle \; = \; \infty \end{aligned}$$

In a real experiment, this would require an arbitrarily large amount of energy to strongly couple to the system.

2.1.1 The density operator

In the real world, practical limitations dictate that physical measurements can come arbitrarily close, yet never achieve, 100% reliability. For example, one can never be sure that a photodetector will inform an experimenter of all of the photons incident to its surface. When such is the case, a classical uncertainty is infused into the system. When an observed system collapses onto a pure quantum state, we the observers may not know which one exactly, and perhaps only the probabilities of each. These situations require the use of a density operator formalism, explored below.

Standard quantum measurement

Elementary quantum mechanics tells us that each observable is represented by an operator, and the outcome of an observation can be any one of the eigenvalues for that operator. The probability of a system $|\psi\rangle$ to be observed with the eigenvalue α, corresponding to the normalized eigenstate $|\psi_\alpha\rangle$, is given by

$$\Pr[\alpha] = \langle\psi_\alpha|\psi\rangle \tag{2.1}$$

If a measured system results in α, then the system is *conditioned* on the result α, and collapses onto a new state described by the unnormalized vector $|\tilde{\psi}^+\rangle$[1] given by

$$|\tilde{\psi}_\alpha^+\rangle = |\psi_\alpha\rangle\langle\psi_\alpha|\psi\rangle = P_\alpha|\psi\rangle \tag{2.2}$$

where

$$P_\alpha = |\psi_\alpha\rangle\langle\psi_\alpha| \tag{2.3}$$

is the projection operator onto the eigenstate $|\psi_\alpha\rangle$. Obviously the normalized version of the new state is

$$|\psi_\alpha^+\rangle = \frac{|\tilde{\psi}_\alpha^+\rangle}{\langle\psi_\alpha|\psi\rangle} = \frac{P_\alpha|\psi\rangle}{\Pr[\alpha]} \tag{2.4}$$

Quantum measurement with density operators

The density operator of a state $|\psi\rangle$ is defined as[2]

$$\rho = |\psi\rangle\langle\psi|$$

In this example the density operator will obviously contain as much information as the original wavefunction $|\psi\rangle$

[1] The tilde ˜ implies an unnormalized vector or operator.

[2] Operators will not necessarily have hats on them, e.g. $\hat{\rho}$; such symbols should have clear meaning from the context.

However, often it is desirable to express a classical uncertainty in which quantum state a system is in, which is achieved by a linear combination of various density operators, each multiplied by a (classical) probability bearing our knowledge for the likelihood for each operator,

$$\rho = \sum_a P(a)|\psi_a\rangle\langle\psi_a| \tag{2.5}$$

where the various wave functions $|\psi_a\rangle$ need not be orthogonal.

The basic quantum measurement process can be alternatively expressed in the notation of density operators. If a measurement is made on a system ρ, then the result α implies the new system is described by

$$\tilde{\rho}_\alpha^+ = P_\alpha \, \rho \, P_\alpha \tag{2.6}$$

and this result will occur with probability

$$\Pr[\alpha] = \mathrm{Tr}\left[\tilde{\rho}_\alpha^+\right] \tag{2.7}$$

The new normalized density operator, conditioned on the result α is given by

$$\rho_\alpha^+ = \frac{\tilde{\rho}_\alpha^+}{\Pr[\alpha]} \tag{2.8}$$

Notice that the probability of an outcome depends on both the classical uncertainty the observer has on the state, as well as the fundamental quantum uncertainty inherent in quantum mechanics.

2.2 Early position measurement theory

In 1987, Caves and Milburn [1] introduced a quantum mechanical model for continuous position measurement by generalizing the formalism known as "operations & effects" theory. Essentially an imprecise measurement will be an 'operation' on the density operator i.e. a super-operator. The 'effect' is to collapse the density operator to a variance determined by the measurement apparatus.

This section follows much of the derivations provided by Caves and Milburn, which sets out the favour of formalism employed by Wiseman in his PhD thesis, and developed in the next section.

2.2.1 The meter

This scheme starts by defining a 'meter' that possesses a minimum uncertainty, which involves restricting the height of the wavefunction describing the meter. By

normalization this implies a minimum width, or minimum uncertainty. A meter is wheeled up to a quantum system which is to be measured, and is allowed to interact with the system. With some assumptions concerning the meter, the 'effect' on the density operator describing the quantum system can then be determined. The meter is wheeled away from the system, and then a measurement is made on the meter, which contains information on the system.

Removing the meter from the system and applying the projection postulate on the meter, one places the 'quantum-to-classical' [1] cut or 'Heisenberg cut' [2] just beyond the meter. By allowing the meter to interact with the observed quantum system, yet making the measurement on the meter, the meter is treated as quantum mechanical system, like the system under observation.

One natural example to consider [1] is a meter, labelled by r and with variables $\bar{x}_r$ and $\bar{p}_r$, that is prepared in the pure Gaussian state given by $|\Upsilon_r\rangle$ with a wave function given by

$$|\Upsilon_r\rangle = \int \frac{1}{\sqrt[4]{\pi\sigma}} e^{-\bar{x}_r^2/2\sigma} |\bar{x}_r\rangle$$

At time $r\tau$, this meter is allowed to interact with the system, which just prior to the interaction is described by $\hat{\rho}^-$ with variables $\hat{\bar{x}}$ and $\hat{\bar{p}}$. The total Hamiltonian first proposed by Von Neumann [3] is of the form

$$\hat{H}_t = \delta(t - r\tau)\hat{x}\hat{\bar{p}}_r$$

where $\hat{x}$ is the position operator of the quantum system, and $\hat{\bar{p}}$ is the momentum operator of the meter. This is an impulsive measurement, where such a small interaction time leads to the individual Hamiltonians for the evolution of the meter and quantum system to be neglected.

Next the meter is removed from the system and an arbitrarily precise measurement on the position of the meter is made. The result of a measurement on the meter is given by the trace over the system variables

$$P(\bar{x}_r) = \mathrm{Tr}\left[\hat{\Upsilon}(\bar{x}_r)\hat{\rho}^-\hat{\Upsilon}^\dagger(\bar{x}_r)\right] \tag{2.9}$$

where $\hat{\Upsilon}(\bar{x}_r)$ is the 'operation', related to the 'meter' by

$$\begin{aligned}\hat{\Upsilon}(\bar{x}_r) &= \langle \bar{x}_r | e^{-i\hat{x}\hat{\bar{p}}_r\hbar} | \Upsilon_r \rangle \\ &= \frac{1}{\sqrt[4]{\pi\sigma}} e^{-(\hat{\bar{x}}_r - \hat{x})^2/2\sigma}\end{aligned}$$

Once the results of a measurement are known, the wavefunction of the quantum system collapses. The 'effect' on the quantum system is a new density operator $\hat{\rho}^+$

conditioned on the results of the measurement, and normalized to

$$\hat{\rho}^{+} = \frac{\hat{\Upsilon}(\bar{x}_r)\hat{\rho}^{-}\hat{\Upsilon}^{\dagger}(\bar{x}_r)}{P(\bar{x}_r)} \tag{2.10}$$

If one takes this measurement model to describe further measurements, two interesting results arise. Firstly, if an initial state of the quantum system considered has a position uncertainty much larger than that of the meter, then by the act of measurement, the system collapses to new wave function with a position uncertainty $\simeq \sigma/2$, defined by the meter. This is to be expected; the results of a measurement should provide the observer with knowledge of a system with an uncertainty defined by the measuring apparatus.

Further more, if many measurements are made so that the quantum state can be described by a stationary set of variances, then the mean momentum undertakes a random walk. This is 'back-action', due to the process of measurement (applying the meter). The random walk in momentum space is unavoidable without some form of feedback, and this is considered by Caves and Milburn later in their paper.

2.2.2 Continuous position measurement

Although Caves and Milburn went on in their paper to describe a process of continuous measurement of position, it is a *non-selective* evolution of the system. That is, knowledge of the system acquired from the process of measurement is not used to condition the density operator describing the system. If one has a *selective*, or conditional, equation for the density operator $\tilde{\rho}_c$, then the 'expectation' or average of operator $\tilde{\rho}_c^{+}$ after a measurement is merely the unconditional evolution $\tilde{\rho}^{+}$. The average is expressed with an expectation operator E[ ], and

$$\mathrm{E}\left[\tilde{\rho}_c^{+}\right] = \tilde{\rho}^{+} \tag{2.11}$$

which is the form of the equation by Caves and Milburn. Thus the results of Caves and Milburn provide a way of checking the correct formulation of a selective process.

Nonselective evolution is not used in feedback processes. Measurements are made and the information derived from the meter are used to feedback into the system to adjust the evolution. Thus a master equation conditioned on measurements is desired, and an excellent derivation is given by Wiseman in his PhD thesis [2].

Non-selective evolution

The non-selective evolution of the density operator of a quantum system under continuous measurement, given by Caves and Milburn, is

$$\begin{aligned}\frac{d\hat{\rho}}{dt} &= -\frac{i}{\hbar}[\hat{H}_0, \hat{\rho}] - \frac{1}{4D}[\hat{x}, [\hat{x}, \hat{\rho}]] \\ &= -\frac{i}{\hbar}[\hat{H}_0, \hat{\rho}] + \frac{1}{4D}\mathcal{D}[x]\rho\end{aligned}$$

where $D = \sigma\tau$ is given by the accuracy of the meter σ and the time between measurement τ, and is a property of type of meter used. $\hat{H}_0$ is the intrinsic dynamics of the system without measurement, and $\mathcal{D}[c]$ is a superoperator defined by

$$2\mathcal{D}[c]\rho = 2c\rho c^\dagger - c^\dagger c\rho - \rho c^\dagger c \tag{2.12}$$

The second term in the master equation should be very familiar; it is used in a master equation describing the decay of atom from an excited state to a ground state. Alternatively, it also describes the loss of light out of an optical cavity, with a decay of the intensity inside the cavity.

This was recognized by the authors, and an analysis highlighted a decay of the off diagonal terms of the density operator in the position basis. This has an obvious explanation; the continuous non-selective *measurement* of position of a system leads to a *collapse* into the position basis, which leads to a *diagonal* representation of the density operator in the position basis.

2.3 Modern measurement theory

The approach to selective measurement theory, taken by Wiseman in his PhD thesis [2] adopts a more general and abstract notation. The motivation to measure position is generalized to an arbitrary operator, representing any observable, and is expressed as a creation operator for the system under observation.

The development of this formalism will closely follow the outline by Wiseman in his PhD, but also draws on inspiration in other papers by Milburn and Wiseman [4], [5] [6]. Also note that $\hbar$ is set equal to 1 for the next two sections.

Most quantum stochastic equations arise by treating a quantum system that is in contact with an infinitely large reservoir with an infinite number of degrees of freedom. Coupling to this bath produces the typical damping effects, such as an excited atom decaying into the ground state, or light leaving a cavity. This 'loss' from a system is a phenomenological consequence of the process of detection; for the detectors considered in this thesis, information on a system will require the removal of some part of it for scrutiny. So it is therefore not surprising that the same results

of quantum measurement theory can be developed by replacing the ensemble of apparatuses with a reservoir, and that reservoir is then probed by the observer for information about the system. These results were also developed in Wiseman's PhD thesis; the reader is deferred to this thesis for this physically insightful interpretation.

2.3.1 Developing the formalism

To form a generalized measurement theory as a master equation, it is instructive to consider the most general form of master equation for a single measurement-related operator c, the Lindblad form,

$$\begin{aligned} \dot{\rho} &= \mathcal{L}\rho && (2.13)\\ &= -i[H,\rho] + \mathcal{D}[c]\rho && (2.14)\\ &= -i[H,\rho] + c\rho c^\dagger - \tfrac{1}{2}cc^\dagger\rho + \tfrac{1}{2}\rho cc^\dagger && (2.15) \end{aligned}$$

where $\mathcal{D}$ is superoperator defined in Eq.2.12, and H is the intrinsic dynamics of the system, and $\mathcal{L}$ is the Liouville superoperator defined by

$$\mathcal{L}\rho = -i[H,\rho] + \mathcal{D}[c]\rho \tag{2.16}$$

Equivalently, if ρ^+ is the new density operator under an infinitesimal evolution of the Lindblad form above,

$$\begin{aligned} \rho^+ &= (1 + dt\ \mathcal{L})\rho && (2.17)\\ &= -i[H,\rho]\ dt + c\rho c^\dagger\ dt - \tfrac{1}{2}cc^\dagger\rho\ dt + \tfrac{1}{2}\rho cc^\dagger\ dt && (2.18) \end{aligned}$$

One point to note here is that the operators c are not unique; a transformation of the form

$$\begin{aligned} c &\rightarrow c + \beta && (2.19)\\ H &\rightarrow H - i\tfrac{1}{2}(\beta^* c - \beta c^\dagger) && (2.20) \end{aligned}$$

results in the same master equation. This transform will be useful later.

Measurement

Now, to generalize the process of a measurement, we use the general measurement operators Ω_α to determine the new unnormalised density operator $\tilde{\rho}_\alpha^+$ conditioned on the result α (see Eq.2.6, Eq.2.10),

$$\begin{aligned} \tilde{\rho}_\alpha^+ &= \Omega_\alpha\,\rho\,\Omega_\alpha^\dagger && (2.21)\\ &= \mathcal{J}\left[\Omega_\alpha\right]\rho && (2.22) \end{aligned}$$

where $\mathcal{J}$ is the superoperator defined by

$$\mathcal{J}[c]\,\rho = c\rho c^\dagger \tag{2.23}$$

This form of measurement operator is evidently Markovian, and these operators only need to satisfy the completeness condition

$$\sum_\alpha \Omega_\alpha \Omega_\alpha^\dagger = 1 \tag{2.24}$$

From the inspection of Eq.2.18, if the operator c was interpreted as the operator for a light field output from a cavity with linewidth γ and internal light field a, such that $c = a\gamma^{\frac{1}{2}}$, then Eq.2.18 would describe the measurement in terms of photodetections of the output light field. Sticking with the notation of c, it can be seen from the expansion one could postulate two measurement operators by

$$\Omega_0 = 1 - (iH + \tfrac{1}{2}c^\dagger c)dt \tag{2.25}$$
$$\Omega_1 = \sqrt{dt}\,c \tag{2.26}$$

Here we can interpret the second measurement operator Ω_1 as the detection of a photon, so that one photon is 'removed' from the system via the action of the annihilation operator c on the system. This would be expressed in density operator notation as

$$\tilde{\rho}_1^+ = \Omega_1\,\rho\,\Omega_1^\dagger = dt\ c\rho c^\dagger \tag{2.27}$$

Notice also that if Ω_1 describes the detection of a a photon, then the probability of detection (See Eq.2.7) is given by

$$\begin{aligned}\Pr[1] &= \mathrm{Tr}\,(\Omega_1\rho\Omega_1)\\ &= \mathrm{Tr}\left(c\rho c^\dagger\right)\ dt\\ &= \langle c^\dagger c\rangle\ dt\end{aligned} \tag{2.28}$$

where we have used, in the last line, the invariance of the trace under cyclic permutations, and the expectation of an operator $\langle A\rangle$ being equal to the trace of the operator by the density matrix $\mathrm{Tr}\,(A\rho)$. Obviously this fits our previous ideas of measurement; the probability of detection over a time will be proportional to the number of photons present and the length of measurement time.

The second measurement operator describes the evolution of the system when no photodetection has occurred.

$$\begin{aligned}\tilde{\rho}_0^+ &= \Omega_0\,\rho\,\Omega_0^\dagger\\ &= \rho - (iH - \tfrac{1}{2}c^\dagger c)\rho\ dt - \rho(-iH - \tfrac{1}{2}c^\dagger c)\ dt\end{aligned}$$

$$\simeq \quad \rho - i[H,\rho]dt - \tfrac{1}{2}(c^\dagger c\rho + \rho c^\dagger c)\ dt$$

to first order in dt.

Both of these processes are purity preserving; if a system is known to be in a pure state, then it remains in a pure state under either measurement evolution.

If the density operator is *not* conditioned on the measurement results, then one can calculate the unconditional, or non-selective evolution of the density operator. If a measurement has been made, but the results not observed, then the system will lie in a mixture of the two states weighted by their respective probabilities (see Eq.2.8)

$$\begin{aligned} \rho^+ &= \sum_{\alpha=0,1} \Pr[\alpha]\rho_\alpha^+ \\ &= \sum_{\alpha=0,1} \tilde{\rho}_\alpha^+ \end{aligned} \tag{2.29}$$

and conveniently, the dynamics of an unobserved system reproduce the original Lindblad dynamics

$$\begin{aligned} \rho^+ &= \sum_{\alpha=0,1} \tilde{\rho}_\alpha^+ \\ &= \rho - i[H,\rho]dt - \tfrac{1}{2}(c^\dagger c\rho + \rho c^\dagger c)\ dt + c\rho c^\dagger\ dt \\ &= \rho - i[H,\rho]dt - \mathcal{D}[c]\rho\ dt \\ &= (1 + dt\,\mathcal{L})\rho \end{aligned} \tag{2.30}$$

Thus we have two measurement operators that relate to the measurement of photons emitted from a quantum system, and completely define any arbitrary evolution of the Lindblad form.

Inefficient detection

What if we have physical inaccuracies in our macroscopic measuring device? The beauty of the density operator formalism is its ability to deal with classical uncertainties, and one of the obvious ways this may enter into a quantum experiment is the inefficient measurement. In the case of photodetections, a photon may only have a *probability* of being detected. Thus the probability of detection needs to be reduced by a fraction η, usually described as a detector efficiency.

We can postulate a new measurement operator for the detection of a photon by

$$\Omega_1 = \sqrt{\eta dt}\, c \tag{2.31}$$

and for a null measure, two operators which need to be summed over

$$\begin{aligned} \Omega_0 &= 1 - (iH + \tfrac{1}{2}c^\dagger c)dt \qquad (2.32) \\ \Omega_0' &= \sqrt{(1-\eta)dt}\; c \qquad (2.33) \end{aligned}$$

The summation is required as the evolution of the density operator will naturally no longer be purity preserving.

The new density operator conditioned on photon detection is

$$\begin{aligned} \tilde{\rho}_1^+ &= \Omega_1 \,\rho\, \Omega_1 \\ &= \eta\, c\rho c^\dagger\, dt \qquad (2.34) \end{aligned}$$

and the new probability of detection $\Pr[1] = \mathrm{Tr}\,(\Omega_1 \rho \Omega_1) = \eta\langle c^\dagger c\rangle\, dt$ is equal to the original probability of occurrence $\langle c^\dagger c\rangle\, dt$, tempered by the detector efficiency η. The density operator conditioned on the null count is now

$$\begin{aligned} \tilde{\rho}_0^+ &= \sum_{\alpha=0}^{0'} \Omega_\alpha \rho \Omega_\alpha \\ &= \big(1 - (iH + \tfrac{1}{2}c^\dagger c)dt\big)\,\rho\,\big(1 - (-iH + \tfrac{1}{2}c^\dagger c dt\big) \\ &\quad +(1-\eta)c\rho c^\dagger\, dt \\ &\simeq \rho - i[H,\rho]\, dt - \tfrac{1}{2}(c^\dagger c\rho + \rho c^\dagger c)\, dt + (1-\eta)c\rho c^\dagger\, dt \qquad (2.35) \end{aligned}$$

to first order in dt, so that the new total unconditioned density operator still obeys the Lindblad form (See Eq.2.30 & Eq.2.29)

$$\begin{aligned} \rho^+ &= \sum_{\alpha=0,1} \tilde{\rho}_\alpha^+ \;=\; \tilde{\rho}_0^+ + \tilde{\rho}_1^+ \\ &= \rho - i[H,\rho]dt - \tfrac{1}{2}(c^\dagger c\rho + \rho c^\dagger c)dt \\ &\quad +(1-\eta)c\rho c^\dagger dt + \eta c\rho c^\dagger dt \\ &= \rho - i[H,\rho]dt - \tfrac{1}{2}(c^\dagger c\rho + \rho c^\dagger c)dt + c\rho c^\dagger dt \\ &= (1 + dt\,\mathcal{L})\rho \qquad (2.36) \end{aligned}$$

2.3.2 Stochastic master equation

Now that we have generalized the formalism to deal with inefficient measures, it is necessary to postulate a relationship between the master equation and the classical, macroscopic process of detection. This is to reformulate the master equation in the form of an explicitly stochastic master equation (SME), which specifies the 'quantum trajectory' of the system.

To do this, we present the measurement results as a random variable dN_c, where

the subscript c implies that the variable is conditioned on the measurement history. One natural presentation is that dN_c is either 0 or 1, conditioned on whether a photon is detected, and

$$\mathrm{E}\left[dN_c\right] = \mathrm{Tr}\left[c^\dagger\, c\, \rho_c\right] \tag{2.37}$$

As an aside, it is also equivalent to postulate dN_c by specifying the expected value and a relationship for all moments of dN_c

$$\begin{aligned} \mathrm{E}\left[dN_c\right] &= \mathrm{Tr}\left[c^\dagger\, c\, \rho_c\right] \\ dN_c &= dN_c^2 \end{aligned}$$

where the 2nd line dictates that the result dN_c can only take values of either 0 or 1. This form of postulation relates to the new stochastic calculus developed by Wiseman, which is more general than the Itó or Stratonovich calculus usually encountered. 'Wiseman' stochastic calculus is required for the development of direct quantum feedback, and this type of feedback will be considered later.

With this new classical measurement variable dN_c, a new *conditional* stochastic master equation for ideal measurement can be expressed

$$d\rho_c = \left\{dN_c\mathcal{G}[c] - dt\,\mathcal{H}[iH + \tfrac{1}{2}c^\dagger c]\right\}\rho_c \tag{2.38}$$

where $\mathcal{G}$ and $\mathcal{H}$ are two new non-linear density operators

$$\mathcal{G}[c]\rho = \frac{c\rho c^\dagger}{\mathrm{Tr}\left[c^\dagger\, c\, \rho\right]} - \rho \tag{2.39}$$

$$\mathcal{H}[c]\rho = c\rho + \rho c^\dagger - \mathrm{Tr}\left[c\rho + \rho c^\dagger\right] \tag{2.40}$$

Three important points arise from this expression. The first is that the evolution for either a count or null result ($dN_c = 1$ or 0 respectively) reduces to the early conditional evolution equations (See Eq.2.34 and Eq.2.35). Also, unconditional evolution for this equation is equivalent to taking the expectation of the equation(See Eq.2.11) so that when dN_c is replaced with $\mathrm{E}\left[dN_c\right] = \mathrm{Tr}\left[c^\dagger\, c\, \rho_c\right]$ the equation reproduces unconditional evolution (See Eq.2.30 & Eq.2.36)

Finally, one can show that this equation is equivalent to a stochastic Schrdinger equation (SSE). This is not surprising, since, for perfect measurement, pure states remain pure, and the evolution of the density operator can be modelled as the evolution of a linear combination of pure states. Each state will evolve by the SSE equivalent to the SME above.

It is easily shown that the equivalent SME for inefficient measurements, using the measurement operators Eq.2.31, Eq.2.32 & Eq.2.33, is

$$d\rho_c = \left\{dN_c\mathcal{G}[\sqrt{\eta}\, c] - dt\,\mathcal{H}[iH + \eta\tfrac{1}{2}c^\dagger c] + dt\,(1-\eta)\mathcal{D}[c]\right\}\rho_c \tag{2.41}$$

where η is the detector efficiency.

2.4 Homodyne detection

Now that a general measurement process has been described for photon detection, it is possible to extend the general operator c to include a local oscillator so that a description of homodyne detection can be developed. One approach used is the transformation invariance of the Liouvillien form of the master equation, to a modification of the operator c and H (See Eq.2.19 & 2.20). One also uses the fact that for homodyne detection we take the limit of a large local oscillator, which means the point process dN_c with Poissonian statistics can be modelled with a random variable δN with Gaussian statistics.

This will be a heuristic walk through the arguments; more rigorous developments have been provided by Carmichael [7], Milburn and Wiseman [5] [4] [6], and Wiseman [2].

By adding a local oscillator (LO), the measurement operators are being changed, and two new measurement operators can be postulated;

$$\Omega_0 = 1 - \left(iH + \tfrac{1}{2}c\beta^* - \tfrac{1}{2}c^\dagger\beta + \tfrac{1}{2}(c^\dagger + \beta^*)(c+\beta)\right) dt \tag{2.42}$$

$$\Omega_1 = \sqrt{dt}\,(c+\beta) \tag{2.43}$$

Let's keep β real for this analysis; it turns out that for homodyne detection this is equivalent to a phase measurement.

Simple homodyne detection

A large coherent field with a photon flux equal to $\frac{\beta^2}{(1-\zeta)}$ is injected into one port of a beam splitter of transmittance ζ. The other port is injected with the output field of the cavity c so that the transmitted beam becomes

$$b = \beta + c \tag{2.44}$$

and perfect measurement of this beam is given by $dN_c = b^\dagger b\, dt$, with an average photon flux given by

$$\mathrm{E}\left[dN_c\right] = \mathrm{Tr}\left[(\beta^2 + \beta(c+c^\dagger) + c^\dagger c)\rho_c\right]\, dt \tag{2.45}$$

Note that the above analysis produces the correct SME

$$d\rho_c = \left\{dN_c\mathcal{G}[c+\beta] + dt\,\mathcal{H}[-iH - \beta c - \tfrac{1}{2}c^\dagger c]\right\}\rho_c \tag{2.46}$$

for the measurement operators defined in Eq.2.42 and E1.2.43. Furthermore, if β is real then this corresponds to homodyne detection on the phase quadrature of the output light field, $c + c^\dagger$.

Now we select a time scale small to the system dynamics but large enough for a high number of photodetections, so that we can model the Poissonian point process dN_c with a normal random variable δN. This time scale is $dt \sim \beta^{-3/2}$, so that in the limit $\beta \to \infty$ the mean photon flux is still large $\sim \beta^2 dt \sim \beta^{1/2}$ and given by Eq.2.45 reducing to

$$\mathrm{E}\left[dN_c\right] \simeq \left(\beta^2 + \beta\langle c + c^\dagger\rangle\right) dt \sim \beta^{1/2} \tag{2.47}$$

For a Poissonian process the variance is equal to the mean $\approx \beta^2\, dt \sim \beta^{1/2}$ and we can model the photodetections statistics described above (dN_c) with the normal variable δN given by

$$\delta N = \beta^2\, dt \left(1 + \frac{\langle c + c^\dagger\rangle}{\beta}\right) + \beta dW \tag{2.48}$$

where dW is the Wiener increment satisfying the Itó relation $\mathrm{E}[dW^2] = dt$

Expanding out the SME (Eq.2.38) to order $\beta^{-1/2}$ by substituting in $dt \sim \beta^{-3/2}$ and $dN_c \to \delta N$, we end up with final result for this chapter

$$d\rho_c = -i\left[H, \rho_c\right]\, dt + \mathcal{D}[c]\rho_c\, dt + dW\, \mathcal{H}[c]\rho_c \tag{2.49}$$

There exists the inefficient measurement SME, which, after the above derivation, has the form

$$d\rho_c = -i\left[H, \rho_c\right]\, dt + \mathcal{D}[c]\rho_c\, dt + \sqrt{\eta}\, dW\, \mathcal{H}[c]\rho_c \tag{2.50}$$

Cleaning up

To express these equations not as function of the light field output, but of the light field inside the cavity we make the substitution $c = a\gamma^{\frac{1}{2}}$. Furthermore, by subtracting off the LO, and then take the limit of a large LO $\beta \to \infty$, the measured number of photodetections can be transformed to a scaled photocurrent I_c by

$$I_c = \lim_{\beta\to\infty} \frac{\delta N - \beta^2 dt}{\beta dt} = \langle c + c^\dagger\rangle\, dt + \xi \tag{2.51}$$

where $\xi = dW/dt$ is a white noise term related to the Wiener process.

Thus if we multiply the operator a by the constant $-i$, the photocurrent becomes

$$I_c = \langle -i(a - a^\dagger)\rangle\, dt + \xi$$

and the current is now proportional to the phase $-i(a - a^\dagger)$. Note here that noting that $\mathcal{D}[a] = \mathcal{D}[-ia]$. Finally, the ubiquitous $\hbar$ is reintroduced into the system.

Thus we have the stochastic master equation for perfect homodyne detection

$$d\rho_c = -\frac{i}{\hbar}\left[H, \rho_c\right]\, dt + \gamma\mathcal{D}[a]\rho_c\, dt + \sqrt{\gamma}\, dW\, \mathcal{H}[-ia]\rho_c \tag{2.52}$$

and a stochastic master equation for imperfect homodyne detection, with detector efficiency η, given by

$$d\rho_c = -\frac{i}{\hbar}\left[H, \rho_c\right]\, dt + \gamma\mathcal{D}[a]\rho_c\, dt + \sqrt{\gamma\eta}\, dW\, \mathcal{H}[-ia]\rho_c \tag{2.53}$$

Both describe the selective evolution of the density operator under continuous phase measurement.

2.5 Conclusion

If the observer chooses to condition the density operator on the information acquired during the measurement process, then this is a *selective* process. Since a measurement on a quantum system is subject to a random outcome, the inclusion of a stochastic term is required. For an idealized setup with perfect measurement, one has complete knowledge of the quantum system, and in this situation the system can be described by a wave vector with a stochastic term. However, for realistic systems, imperfect detection and incomplete knowledge of the system, a density operator is again required to describe the system, with a stochastic term expressing the information available from the measurement. Thus, the evolution of the system is stochastic master equation.

The arrival at a SME is the most important outcome of this chapter. A master equation formalism is important for coping with classical uncertainties arising from inefficient measurements. The stochastic element of a master equation is a unique instalment; it is not a typical stochastic element which usually involves undesirable noise sources. It possess real information about the system evolution, and is received through some classical measuring device, the standard quantum optical tool being the homodyne setup. The stochastic master equation Eq. 2.53 describes the evolution of a quantum system under measurement, by involving a stochastic element which will depend on the measurement results of each run, or 'quantum trajectory'.

Chapter 3

Quantum Systems

One the process of (homodyne) measurement is understood, the next step is to find quantum systems to measure. In this case, both the flexible mirror and an atom inside the measurement cavity provide good candidates. Their Hamiltonians both have the correct form, and in the case of the flexible mirror, readily available experimentally.

One the other hand, trapping an atom inside a cavity is not so trivial, and the work by Hood, Lynn, Doherty, Parkins and Kimble at the Californian Institute of Technology have made remarkable progress into this field [8]. Here it is interesting to note that the atom is both trapped and measured by the same light field, where the experiment is operating in the strong coupling regime. Whether the theory spelt out below is applicable in their situation is unclear.

In this chapter, the atom dynamics are discussed, and with suitable approximations, a Hamiltonian is introduced. This Hamiltonian is used in conjunction with the equation for homodyne measurement from the last chapter, and, with further simplifications, a measurement equation for the atomic position is presented. Finally, choosing an initial gaussian state and harmonically confining the atom, reduces the total description of the quantum state to the means and covariances of $\langle x \rangle$ and $\langle p \rangle$.

3.1 Measuring the atom position

From the previous chapter we have an equation for the evolution of the density operator for continuous measurement.

$$d\rho_c = -\frac{i}{\hbar}\left[H, \rho_c\right]\, dt + \gamma \mathcal{D}[a]\rho_c\, dt + \sqrt{\gamma\eta}\, dW\, \mathcal{H}[-ia]\rho_c \tag{3.1}$$

we need to specify the form of the internal dynamics of the system under observation inside the cavity, namely H. To do this we need to select a system we wish to observe.

Here two closely related systems can be modelled; a harmonically bound atom inside the cavity, or a flexible mirror making up one end of the cavity. Though both reduce to the same simple Hamiltonian form under certain approximation, the

case of the harmonically bound atom will be look at here. It is easy to see that for a flexible mirror with a linear restoring force proportional to position, along the lines of Hooks law, that the same mechanical dynamics will be observed as for a harmonically bound atom; namely quantized simple harmonic motion.

3.2 An atom inside a cavity

Let now assume a two level atom is inside a cavity interaction with the quantized field inside. First we make the dipole approximation, that the atom is much smaller in dimensions than the wavelength of the light field. Secondly, we make the rotating wave approximation, so that we can ignore non-energy conserving dynamics that occur much on much smaller time scales than we are considering here.

Then we can start with a model of a two level atom interacting with a single mode of the quantized field. In the Heisenberg picture this is a Hamiltonian of the form [9]

$$H = \hbar\omega a^\dagger a + \hbar\omega_0\sigma_z + \hbar(g\sigma_- a^\dagger + g^*\sigma_+ a)\cos(k_0 x) \tag{3.2}$$

where a and $a^\dagger$ are the bosonic creation and annihilation operators for the quantized light field inside the cavity, σ_z, σ_- and σ_+ are the pseudo spin operators for the atom[1], g is the dipole coupling constant and k_0 the wavenumber for the field.

If we assume that the light field is sufficiently far detuned from the resonance frequency of the atom, with a detuning Δ, then spontaneous emission can be ignored, Then taking the equation above, the upper level can be adiabatically eliminated (See [9], Appendix 17.A) and the resulting effective Hamiltonian, in the interaction picture for the cavity, becomes

$$H_{eff} = \hbar\Delta\sigma_z + \frac{2\hbar g^2}{\Delta}\sigma_z a^\dagger a\cos^2(k_0 x) \tag{3.3}$$

Now we make the further approximation that the atom is positioned very close to the anti-node of the standing optical wave inside the cavity, so that $\cos^2(k_0 x)$ can be very closely approximated as linear in x. Furthermore, if the atoms start in the ground state, that is where they'll remain, and the first term becomes a constant which does not effect the dynamics of the system, and can be ignored, while $2\sigma_z \to 1$(See footnote), so the interaction Hamiltonian becomes of the form

$$H'_{eff} = \frac{\hbar k_0 g^2}{\Delta}a^\dagger a\, x \tag{3.4}$$

Now, if we add to the Hamiltonian the dynamics of the mechanical motion of the atom H_m and a coherent driving term for the coupling of the light field out of

[1] $\sigma_z = \frac{1}{2}|1\rangle\langle 1| - \frac{1}{2}|0\rangle\langle 0|$, $\sigma_- = |0\rangle\langle 1|$ and $\sigma_+ = |1\rangle\langle 0|$

the cavity $H_d$we have the Hamiltonian given by Doherty and Jacobs [10]

$$H = H_m - \frac{\hbar k_0 g^2}{\Delta} a^\dagger a \, x + H_d \tag{3.5}$$

where H_d is of the form

$$H_d = i\hbar E(a - a^\dagger) \tag{3.6}$$

where E is related to the power of the laser P by $E = \sqrt{\gamma P/(\hbar\omega_0)}$, γ is the decay rate of the cavity and ω_0 is the angular frequency of the cavity field.

Note the form of the Hamiltonian, being in the interaction picture, which would be the same for the flexible mirror. The term $a^\dagger a \, x$ is the correct form necessary since $a^\dagger a$ is the phase generator for light, which is now proportional to position x, so that homodyning will produce position measurement.

A pleasantly simple way to look at this sort of Hamiltonian is to imagine that the atom merely introduces a change in refractive index, which is linear if the atom is confined to part of the optical standing wave which is also linear in strength with respect to position. A refractive index change can be noticed as a phase shift of light.

For the mirror, it is well known that the phase of light off a cavity undergoes a change from positive to negative as a mirror moves through the resonance condition. On resonance, this change in phase is approximately linear with position, and hence phase measurement will provide position information.

3.2.1 Measurement of atomic position

Now we can incorporate Eq.2.40 for the dynamics of the atom with Eq.3.1 describing the homodyne measurement process. By taking the limit of large γ, or a very lossy cavity, we are getting a very good measure of position, since the light field interacting with the atom, and containing position information, leaves the cavity quickly to avail itself to the measurement process. Thus the cavity mode dynamics described by a are 'slaved' to the atom dynamics, and the operators a and $a^\dagger$ can be eliminated adiabatically. Below is another heuristic walk-through of the rigorous analysis given by Doherty and Jacobs [10]

Eliminating the cavity modes

If we apply the displacement operator $D(-\alpha)$ to the density operator for the atom/cavity system, where $\alpha = -2E/\gamma$ is the steady state of the cavity without the atom, then the transformation

$$\rho' = D(-\alpha)\rho D^\dagger(-\alpha) \tag{3.7}$$

produces the equation for the atom/cavity system

$$\begin{aligned} d\rho' &= -\frac{i}{\hbar}\left[H_m - \hbar g(a^\dagger a + \alpha(a+a^\dagger) + |\alpha|^2)x, \rho_c\right]\, dt \\ &\quad +\gamma\mathcal{D}[a]\rho_c\, dt + \sqrt{\gamma\eta}\, dW\, \mathcal{H}[-ia]\rho_c \end{aligned} \tag{3.8}$$

where for the atom

$$g = \frac{k_0 g_0^2}{\Delta}$$

This centres the density operator around the vacuum state for the cavity, where it will be easy to make further approximations.

For adiabatic elimination, the rate of dynamics of the atom motion must be slower than the decay rate, so that information about the atom motion is 'up-to-date'. Mathematically the requirement becomes

$$\left|\frac{\langle H_m\rangle}{\gamma}\right| \sim \epsilon \ll 1 \tag{3.9}$$

where ϵ is a small parameter governing the approximation.

A coherent state, centered around the vacuum state by the density operator, still possess many Fock states with which the atom can interact with. The strength of these Fock states will decrease the further from the vacuum state they are, and the interaction with the atom will naturally decrease as well, and it is appropriate to make the approximation that the various ρ' elements in the number basis scale as $\rho'_{mn} \propto \epsilon^{(m+n)}$.

If the atom/cavity operator is expanded out over the various Fock states around the vacuum state, then

$$\begin{aligned} \rho' &= \rho^a_{00}|0\rangle\langle 0| + (\rho^a_{10}|1\rangle\langle 0| + H.c.) \\ &\quad +\rho^a_{11}|1\rangle\langle 1| + (\rho^a_{20}|2\rangle\langle 0| + H.c.) + O(\epsilon^3) \end{aligned} \tag{3.10}$$

and we are now interested in the atom dynamics, tracing over the cavity modes

$$\rho^a = \mathrm{Tr}_c[\rho'] = \rho^a_{00} + \rho^a_{11} + O(\epsilon^3) \tag{3.11}$$

Substituting the above expansion into the SME for homodyne measurement (Eq.3.1) produces many coupled stochastic equations for the various elements of ρ'; for example

$$\begin{aligned} d\rho^a_{00} &= -\frac{i}{\hbar}\left[H - \hbar g|\alpha|^2 x, \rho^a_{00}\right]\, dt + ig\alpha(x\rho^a_{10} - \rho^{1\dagger}_{10}x)dt + \gamma\rho^a_{11}dt \\ &\quad -i\sqrt{\eta\gamma}(\rho^a_{10} - \rho^{a\dagger}_{10} - \mathrm{Tr}[\rho^a_{10} - \rho^{a\dagger}_{10}]\rho^a_{00})dW \end{aligned} \tag{3.12}$$

The next analytical step requires quite a few proposals. The off-diagonal elements need to be expressed in terms of the diagonal elements, and to complicate matters, all elements are being driven by a stochastic white noise dW. However, continuing to use the adiabatic approximation, it can be seen that most off diagonal terms will be strongly damped, and steady state values can be assumed. These steady values can be expressed in terms of the diagonal elements, and the diagonal elements can be added together to arrive at an equation for the motion of the atom under homodyne detection

$$d\rho = -\frac{i}{\hbar}\left[H_m - \hbar g|\alpha|^2 x, \rho\right]\, dt + 2k\mathcal{D}[x]\rho\, dt + \sqrt{2k\eta}\, dW\, \mathcal{H}[x]\rho \tag{3.13}$$

where

$$k = \frac{2k_0^2 g_0^4 |\alpha|^2}{\gamma\Delta^2} \tag{3.14}$$

may be referred to as a coupling constant, as is the rate at which measurement information is obtained from the system, and consequently the rate at which noise drive the momentum, due to back action.

The second term in the commutation brackets is merely the dipole force on the atom, or the in the case of the mirror, the radiation force. As with Doherty and Jacobs, this force will be ignored, as it could be cancelled with an appropriate classical linear force in the opposite direction, by adding a term to the Hamiltonian H_m.

3.2.2 Determining mean and covariances

The equation developed so far is very close to what is required to apply the engineering concepts of control theory; what is required is that the quantum state of the system is describe by only the means and covariances for x and p.

This can be achieved if the Hamiltonian for the mechanical motion of the particle H_m is no greater than 2nd order in both position and momentum. Under this condition, the system state will remain gaussian as it evolves, provided it starts in a gaussian state [11]. This presumption of beginning in a gaussian state is quite reasonable, as it is believed that any non-classical state will rapidly evolve to gaussian state.

Provided that a.) the state start in a gaussian state and b.) the Hamiltonian for mechanical motion is of no greater order than 2 in position and momentum, then the quantum state will be specified by means and covariances for all time. Calculating these amounts to calculating the dynamics of the stochastic system.

Means

The means of x and p will be denoted as $\langle x\rangle$ and $\langle p\rangle$, and are easy to calculate from the expectation of the operators by

$$\langle c\rangle = \mathrm{Tr}[c\,\rho] \tag{3.15}$$

and

$$d\langle c\rangle = \mathrm{Tr}[c\,d\rho] \tag{3.16}$$

Using the these definitions, and invariant property of the trace under cyclic permutations of the operators within, it is trivial to arrive at

$$d\langle x\rangle = -\frac{i}{\hbar}\langle[x, H_m]\rangle dt + 2\sqrt{2\eta k}\,V_x\,dW \tag{3.17}$$

$$d\langle p\rangle = -\frac{i}{\hbar}\langle[p, H_m]\rangle dt + 2\sqrt{2\eta k}\,C\,dW \tag{3.18}$$

where $V = \langle x^2\rangle - \langle x\rangle^2$ and C is the symmetric covariance $C = \frac{1}{2}\langle xp + px\rangle - \langle x\rangle\langle p\rangle$ Note that no assumption was made regarding the Gaussian nature of the state; the equation hold for any state of the system.

Covariances

Calculating the variances and covariances are not as trivial as for the means. If the change in the variance V_x is given by

$$dV_c = d\langle x^2\rangle - d\langle x\rangle^2$$

it is easy to see $d\langle x^2\rangle$ will be given by $\mathrm{Tr}[x^2 d\rho]$, but how does one determine the differential of the square of stochastic variable $\langle x\rangle$?

Here one turns to the rules of Itó calculus. If X is a stochastic variable given by

$$X = f\,dt + g\,dW$$

then for a function $F(X)$ of X, the differential is given by

$$dF = \tilde{f}\,dt + \tilde{g}\,dW$$

where

$$\tilde{f} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial X}f + \tfrac{1}{2}\frac{\partial^2 F}{\partial X^2}\,g^2$$

$$\tilde{g} = \frac{\partial F}{\partial X}\,g$$

Here the stochastic terms dW cancel from $d\langle x^2\rangle$ and $d\langle x\rangle^2$, and the resulting differentials can be divided through by dt to present

$$\dot{V}_x = -\frac{i}{\hbar}\langle[x^2, H_m]\rangle + \frac{2i}{\hbar}\langle x\rangle\langle[x, H_m]\rangle - 8\eta k V_x^2 \tag{3.19}$$

$$\dot{V}_p = -\frac{i}{\hbar}\langle[p^2, H_m]\rangle + \frac{2i}{\hbar}\langle p\rangle\langle[p, H_m]\rangle + 2k\hbar^2 - 8\eta k C^2 \tag{3.20}$$

$$\begin{aligned}\dot{C} = &-\frac{i}{2\hbar}\langle[xp + px, H_m]\rangle + \frac{i}{\hbar}\langle x\rangle\langle[p, H_m]\rangle \\ &+\frac{i}{\hbar}\langle p\rangle\langle[x, H_m]\rangle - 8\eta k V_x C\end{aligned} \tag{3.21}$$

Notice to this stage H_m hasn't yet been specified.

Harmonically bound

Now it is pertinent to assume some form of mechanical restraint. One natural assumption is harmonically binding the atom to the anti-node of the light field. This could be achieved with the dipole force from a 2nd light field. It is at this point that the theory presented thus far differs from the experiment setup by Hood *et. al*, but it is certainly worth considering how this theory would need to be modified to be be experimentally relevant to Hood. There is also scope for considering other restraints, such as the wavelength separation between the measuring and containment beam [12].

With the assumption

$$H_m = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} \tag{3.22}$$

it is obvious that the means reduce to

$$d\langle x\rangle = \frac{\langle p\rangle}{m}dt + 2\sqrt{2\eta k}\, V_x\, dW \tag{3.23}$$

$$d\langle p\rangle = -m\omega^2\langle x\rangle dt + 2\sqrt{2\eta k}\, C\, dW \tag{3.24}$$

while the variances reduce to

$$\dot{V}_x = 2C/m - 8\eta k V_x^2 \tag{3.25}$$

$$\dot{V}_p = -m\omega^2 C + 2k\hbar^2 - 8\eta k C^2 \tag{3.26}$$

$$\dot{C} = V_p/m - m\omega^2 V_x - 8\eta k V_x C \tag{3.27}$$

Here it is obvious that the means and covariances have decoupled, which is an important point in control theory, termed *separation*. Furthermore, the covariances don't posses any random nature to them, they evolve along with the means, and end up in a steady state. This property will be utilized in the simulation by assuming

the steady state given by

$$V_x = \left(\frac{\hbar}{\sqrt{2\eta}m\omega}\right)\frac{1}{\sqrt{\zeta+1}} \tag{3.28}$$

$$V_p = \left(\frac{\hbar m\omega}{\sqrt{2\eta}}\right)\frac{\zeta}{\sqrt{\zeta+1}} \tag{3.29}$$

$$C = \left(\frac{\hbar}{2\sqrt{\eta}}\right)\frac{\sqrt{\zeta-1}}{\sqrt{\zeta+1}} \tag{3.30}$$

where

$$\zeta = \sqrt{1+\frac{16\hbar\eta k^2}{m^2\omega^4}}$$

It's convenient to point out that the Heisenberg uncertainty principle is still obeyed in the evolution of these variances; as the measurement strength k is changed, the product V_xV_p remains greater than $\hbar$.

The means obey some convenient properties, namely being Markovian, linear and with a gaussian noise source described by dW. Being linear with gaussian noise allows one to utilized the mathematics of control theory to present the optimal control process, once a parameter was been designated to be optimized over

3.3 Conclusion

Starting with the stochastic master equation describing continuous measurement, and introducing a Hamiltonian which possess the property of generating a phase shift proportional to the position of a quantum object (either atom or mirror), in the limit of a lossy cavity the motion of the quantum object can be adiabatically slaved to the cavity field, so that measurement of the field relates to a measure of position. The resulting stochastic master equation for position maintains a gaussian state over evolution, provided the Hamiltonian for mechanical motion is no greater than 2nd order in position and momentum. Provided this is so, the state of the quantum object can be describe by 5 parameters, being the mean and variance of the position and momentum, as well as a covariance between the two. Thus it is natural to harmonically confine the quantum object, assume an initial gaussian state, and determine the evolution of the means and covariances for the state. This completely defines the state for all time, and the resulting equations for the means describe a linear Markovian process with gaussian noise, which places the system in a position to be considered by control theory.

Chapter 4

Control Theory

This chapter introduces the ideas of standard control theory. This covers the L*inear* Q*uadratic* R*egualtor* (LQR) and L*inear* Q*uadratic* G*aussian* (LQG) formulation of a control problem. These problems differ, as an LQR is when there is deterministic observation and prefect detection, while an LQG system is one with stochastic noise in both the system evolution and the measurement signal. Both have associated optimal control solutions, which provide the best control to apply to a system to minimize a 'cost function', which relates to improved performance. The optimal control for the LQR is to feedback a linear function of the system state. For the LQG problem the optimal control solution is provided by the Kalman filter, the best estimate of the current state of the system, and the control is the same linear function as the LQR, but of the best state estimate. Thus the LQG solution is often called 'state estimation.'

This chapter begins with the concept of a cost function as a way a to measure the performance of a control policy, as well as determining the state which the control designer wishes to drive, or *regulate*, the system to. Once convenient form of a cost function is one quadratic in both the state vector and control vector. Next, the principle of optimality connects the desired outcome defined by the cost function with the optimal control policy to implement. A general derivation for an arbitrary cost and system dynamics is illustrated; next it is applied a linear system with a quadratic cost, the LQR problem. Finally, noise is introduced into both the system and the measurement process, the LQG model, and the best-estimate of the system state is given by the Kalman filter. The previous ideas of performance and optimality are reviewed, and the Certainty-Equivalence Principle is expressed, which presents the optimal control solution for the LQG

It will turn out that for the linear systems with gaussian noise previously considered, coupled with a quadratic cost function, the optimal control policy will be to apply the Kalman filter to provide the best estimate of the state of the system.

Note that throughout this chapter the notation will swap from continuous expressions e.g $\mathbf{r}(t)$ to discrete e.g. $\mathbf{r}_t$, and back again. This freedom provides an access to an easy proofs in one form that may not be so easy in the other.

4.1 The cost function

Defining the cost function turns out to be a very important processes in designing a control system. The art of selecting a cost function is an expression of the control designer of the most desired state of the system, as well as the performance of the control policy.

4.1.1 Selecting a cost function

The cost function will express the way a control designer wishes to drive a system by associating each state with a numerical evaluation of desirability. Mathematically, it is a function of both the state of the system $\mathbf{r}$ and the control policy applied to it $\mathbf{u}$, and denoted $c(\mathbf{r}, \mathbf{u})$. By intelligently selecting a mapping of all possible system states and controls onto the positive reals

$$c(\mathbf{r}, \mathbf{u}) : \mathbf{r}, \mathbf{u} \mapsto \mathbb{R}^+ \tag{4.1}$$

a control designer creates a way in which to calculate the desirability of the various states of a system, as well as a measure of the controls used. By mapping onto the reals, is a positive bilinear function.

Cost as a measure of state desirability

It is mathematically convenient to map the most desired state, or state that the controller is trying to regulate to, onto the origin. By doing so the vector $\mathbf{r}$ now describes the difference between the most desired and actual state. Thus the control designer associates a small vector $\mathbf{r}$ with a highly desirable state.

Measure of control strength

A cost function is designed to measure the amount of effort required to regulate to the most desired state, by taking into account the control $\mathbf{u}$ applied. If was done, the naturally unrealistic solution to a control problem would be to apply an arbitrarily large (infinite) feedback, forcing the system to be in exactly the right state. Apart from the impracticalities of such a setup, there are inherent problems with control policies of arbitrary strength. For an arbitrarily large feedback term and a slight time delay in the control loop, oscillations around the origin will arise naturally. Thus a measure of control strength should be implicit in the cost function.

4.1.2 Performance

A cost function $c(\mathbf{r}, \mathbf{u})$ also has a second purpose of providing a criteria for evaluated the performance of a control policy. If one control policy is more likely to produce

states with larger costs (which equates to larger deviations from the desired state) than another control policy, then the latter control policy would be more desirable, and this control policy is said to have better performance. If we denote I as the performance, then smaller numerical values of I implies better performance.

A suitable choice for the performance I is the total amount of cost incurred by a certain control policy. This is the sum (or integral, in continuous time) of the costs, evaluated over the length of time that the control policy is implemented. In discrete time this is seen as

$$I = \sum_t c(\mathbf{r}_t, \mathbf{u}_t, t) \tag{4.2}$$

while the performance criteria expressed in continuous time is

$$I = \int c\left(\mathbf{r}(t), \mathbf{u}(t), t\right)\, dt \tag{4.3}$$

Accurate measure of performance

A second desirable quality in a cost function is that it is a monotonically increasing function as the performance of the control policy decreases. For example, assume that the cost function was given as the numerical linear difference between the actual and desired state variables, i.e. $I = \int \mathbf{r}\, dt$. Then if oscillations around the desired state are set up in the system, the total integrated cost function may end averaging close to zero, which would imply a spuriously good performance. Furthermore, larger oscillations may not relate to a larger performance criterion (and lower performance), and in this situation the cost function is not a good representation of the performance of a system.

4.1.3 A good cost

After elucidating the importance of the cost function, and acknowledging that it is desirable for a cost function to posses a few simple characteristics, it must be recognized that there still exists an infinite number of possible choices. However, analytical easy will lead to a cost function quadratic in both $\mathbf{r}$ and $\mathbf{u}$.

Quadratic solution

One obvious solution is to take the next natural step and square the difference vector $\mathbf{r}$. One could do this via $\mathbf{r}'\mathbf{r}$, or use a more generalized form $\mathbf{r}'\mathbf{P}\mathbf{r}$, where squares are weighted by the diagonal elements of the matrix $\mathbf{P}$, as well as correlations between certain state parameters by the off diagonal terms. To include the control strengths one can add a similar term $\mathbf{u}'\mathbf{Q}\mathbf{u}$, so that the cost function is

$$c(\mathbf{r}, \mathbf{u}) = \mathbf{r}'\mathbf{P}\mathbf{r} + \mathbf{u}'\mathbf{Q}\mathbf{u} \tag{4.4}$$

Since the cost function was defined earlier by Eq.4.1, the matrices $\mathbf{P}$ and $\mathbf{Q}$ must both be positive definite, so that

$$\mathbf{x}'\mathbf{P}\mathbf{x}, \mathbf{x}'\mathbf{Q}\mathbf{x} \geqslant 0 \ , \quad \forall \mathbf{x} \tag{4.5}$$

This solution has further merits. Not only is it quadratic, but its analytical nature lends itself to easily implemented control policies.

The performance criteria become, for both the continuous and discrete time,are (See Eq.4.2 and Eq. 4.3)

$$I = \int c(\mathbf{r}(t), \mathbf{u}(t), t)\, dt = \int \mathbf{r}'(t)\mathbf{P}\mathbf{r}(t) + \mathbf{u}(t)\mathbf{Q}\mathbf{u}(t) \tag{4.6}$$

and

$$I = \sum_t c(\mathbf{r}_t, \mathbf{u}_t, t) = \sum_t \mathbf{r}'_t\mathbf{P}\mathbf{r}_t + \mathbf{u}'_t\mathbf{Q}\mathbf{u}_t \tag{4.7}$$

There are a few further mathematically requirements that allow exact optimal controls to be easily formulated. One of these is that the cost is positive definite in $\mathbf{u}$,which is ensured by Eq.4.5 and Eq.4.1, so that all controls produce an increase in cost. Another is that the cost function is *decomposable*, so that the expected cost at any time $c(\mathbf{r}, \mathbf{u})$ is only a function of the state $\mathbf{r}$, the control $\mathbf{u}$ and t. This is also implied by Eq.4.2 and Eq.4.3.

Once an appropriate cost function has been determined, it is then a process of optimization to determine the optimal control policy to implement.

4.2 Optimal control

The theory of optimal control has a powerful structure which is independent of the form of problem which it is applied to, provided the system posses some simple structure properties. These are that the system is Markovian, the state $\mathbf{r}_t$ of the system is observable at time t, and the expected cost is decomposable (See previous section).

For any stochastic optimization problem, the optimal control policy will have a recursive nature. This will lend itself to easy implementation, especially with the use of digital or analogue signal processing.

For the first part of this section, it will be convenient to express the dynamical equations for the system, the Principle of Optimality, and the optimal control solution, in discrete time. Later the continuous time equations can be derived by taking the limit of infinitesimal displacements.

4.2.1 Principle of Optimality

In other words, one could express the Principle of Optimality by assuming the optimal control exists, and asking what properties it may posses. Below is an alternative expression of the Principle of Optimality in words.

Principle of Optimality an optimal control sequence has the property that, whatever the initial state and the first optimal control may be, the remaining controls also constitute an optimal sequence, if one starts from the state *resulting* from the first control[1]

In other words, say $\mathbf{u}_0(t), t \in (0, T)$ is an optimal control policy starting from state $\mathbf{r}$ and optimized over time T. If, during the application of $\mathbf{u}_0$, the system passes through the state $\mathbf{r}'$ at time $t = \Delta$, then the remaining control $\mathbf{u}_0(t), t \in (\Delta, T)$ itself constitutes new optimal control policy $\mathbf{u}'_0$ $t \in (\Delta, T)$, designed to start from state $\mathbf{r}'$ and optimized over time $T - \Delta$.

4.2.2 Deriving the continuous Optimal Programming equation

Here is a derivation of the Optimal Programming equation in the continuous, completely determined (perfect detection with no stochastic element) case. Both the form of the cost and the evolution of the system state has been generalized to highlight its applicability to any situation. The steps outlined below are based on those taken by Jacobs [13].

Begin by restating the problem in a general form, so that the state evolves as

$$\mathrm{d}\mathbf{r} = \mathbf{g}(\mathbf{r}, \mathbf{u})\, dt \tag{4.8}$$

and the performance criteria I is a function of a general cost $L(\mathbf{r}, \mathbf{u})\, dt$,

$$I(\mathbf{r}, T) = \int_0^T L(\mathbf{r}, \mathbf{u})\, dt \tag{4.9}$$

Here the performance criterion is expressed explicitly as a function of both the initial state $\mathbf{r}$ and the time T over which the control is to be optimized. We can then split up an arbitrary non-optimal performance criterion into two parts

$$I(\mathbf{r},T) = \int_0^\Delta L(\mathbf{r}, \mathbf{u})dt \; + \; \int_\Delta^T L(\mathbf{r}, \mathbf{u})dt \tag{4.10}$$

[1] quote from Jacobs [13] p142

Here we note that the second term is just another (non-optimal) performance criterion I' with a different initial state $\mathbf{r}'$ and evaluated over a shorter time $T - \Delta$, which leads to

$$I(\mathbf{r},T) = \int_0^{\Delta} L(\mathbf{r},\mathbf{u})dt \ + \ I'(\mathbf{r}', T-\Delta) \tag{4.11}$$

By taking Δ to be infinitesimally small the first term can be expressed

$$\int_0^{\Delta} L(\mathbf{r},\mathbf{u})dt \simeq L(\mathbf{r},\mathbf{u})\Delta$$

while the state $\mathbf{r}'$ can be expressed using Eq.4.8 describing the evolution of the system as

$$\mathbf{r}'(t=\Delta) \simeq \mathbf{r}(t=0) + \mathbf{g}(\mathbf{r},\mathbf{u})\Delta$$

leaving us with

$$I(\mathbf{r},T) \simeq L(\mathbf{r},\mathbf{u})\Delta + I'(\mathbf{r} + \mathbf{g}(\mathbf{r},\mathbf{u})\Delta,\ T-\Delta) \tag{4.12}$$

This can be further approximated with a first-order Taylor series expansion of the second term. With the notation

$$\frac{\partial I'(\mathbf{r},T)}{\partial \mathbf{r}} \doteq \begin{pmatrix} \frac{\partial I'}{\partial r_1} \\ \frac{\partial I'}{\partial r_2} \\ \vdots \\ \frac{\partial I'}{\partial r_N} \end{pmatrix}$$

and N the dimensionality of $\mathbf{r}$

$$I(\mathbf{r},T) \simeq L(\mathbf{r},\mathbf{u})\Delta \ + I'(\mathbf{r},T) + \frac{\partial I'(\mathbf{r},T)}{\partial \mathbf{r}} \cdot \mathbf{g}(\mathbf{r},\mathbf{u})\Delta - \frac{\partial I'(\mathbf{r},T)}{\partial T}\Delta \tag{4.13}$$

Next minimize this equation for the performance criteria with respect to the control $\mathbf{u}$. Here the Principle of Optimality is utilized by equating $I(\mathbf{r},T)$ and $I'(\mathbf{r},T)$, since these will become the same optimal performance criteria, starting from $\mathbf{r}$ and evaluated over T. Replacing both $I(\mathbf{r},T)$ and $I'(\mathbf{r},T)$ with notation for the optimal control $I_0(\mathbf{r},T)$, which leaves

$$I_0(\mathbf{r},T) \simeq \min_u \left\{ L(\mathbf{r},\mathbf{u}) + \frac{\partial I'(\mathbf{r},T)}{\partial \mathbf{r}} \cdot \mathbf{g}(\mathbf{r},\mathbf{u}) \right\} \Delta + I_0(\mathbf{r},T) - \frac{\partial I_0(\mathbf{r},T)}{\partial T}\Delta \tag{4.14}$$

or

$$\frac{\partial I_0(\mathbf{r},T)}{\partial T}\Delta \simeq \min_u \left\{ L(\mathbf{r},\mathbf{u}) + \frac{\partial I_0(\mathbf{r},T)}{\partial \mathbf{r}} \cdot \mathbf{g}(\mathbf{r},\mathbf{u}) \right\} \Delta \tag{4.15}$$

In the limit $\Delta \to 0$ the approximation becomes an equality, and so dividing by Δ,

and removing the functional dependences presents the desired result

$$\frac{\partial I_0}{\partial T} = \min_u \left\{ L(\mathbf{r}, \mathbf{u}) + \frac{\partial I_0}{\partial \mathbf{r}} \cdot \mathbf{g}(\mathbf{r}, \mathbf{u}) \right\} \tag{4.16}$$

Minimizing with respect to $\mathbf{u}$ produces the optimal control $\mathbf{u_0}$

4.3 The LQR

The LQR stands for the L*inear* Q*uadratic* R*egulator*, which is a set of linear equations that describe a system with no noise introduced, coupled with a quadratic cost function. It is a relatively simple situation, and the optimal control solution remains easily accessible.

4.3.1 System description

Most of the systems of interest posses a few common characteristics, which will be assumed true so that the control theory from this chapter can be applied. One of the first is the Markovian nature of the system, in which knowledge of only the present determines the future. As elucidated by Gardiner, true Markovian systems rarely exist in reality. [2] However, Markovian processes with continuous sample paths do exist mathematically and are useful in describing reality.

Another accepted notion about the system is that evolution responds linearly to the feedback signal. These systems are a subset of the more general class of stochastic control problems which won't be considered. Finally, we model a linear system, which implies that the changes in the state depend linearly on the previous states of the system. Often the physical models studied in physics are linear in nature, and thankfully this often allows an exact solution of the evolution of the system. Furthermore, quantum mechanics is a naturally linear theory, and although the measurement processes introduced earlier have a non-linear nature, the system under study results in linear expression of evolution. (See Eq.3.23 and Eq.3.24) As will be seen in the next chapter, it will also be necessary to formulate the feedback mechanism so the system responds in a linear fashion

We can mathematically represent these assumptions with equations in the continuous form

$$\mathrm{d}\mathbf{r} = \mathbf{A}\mathbf{r}\ dt + \mathbf{B}\mathbf{u}\ dt$$

[2] Most systems exhibit a *memory time-scale*, during which the evolution of the systems depends on history, and after which the system is independent of the history, or Markovian. If one makes measurements over time scales larger than this system memory time scale, then the system is Markovian, yet the system then becomes discontinuous over sample paths. This raises the question of whether Markov processes with continuous sample paths actually exist in reality.

The linear nature is obvious from the fact that the change in the system vector $d\mathbf{r}$ is linearly related to the state $\mathbf{r}$ via $\mathbf{A}$, and similarly a linear response to the feedback signal $\mathbf{u}$ with $\mathbf{B}$. [3]

4.3.2 Optimality applied to the LQR

Now it is possible to apply the results in section 4.2.2 for deriving the continuous Optimal Programming equation to a system which is linear in both $\mathbf{r}$ and $\mathbf{u}$, and a cost function I which is quadratic in both $\mathbf{r}$ and $\mathbf{u}$. Both justify the terminology of a Linear Quadratic Regulator.

Restating the problem, we have a system whose evolution is described by the deterministic equation

$$d\mathbf{r} = \mathbf{A}\mathbf{r}dt + \mathbf{B}\mathbf{u}dt \tag{4.17}$$

and we desire to determine the optimal control $\mathbf{u}_0$ to apply to the system to minimize the performance criterion

$$I = \int (\mathbf{r}'\mathbf{P}\mathbf{r} + \mathbf{u}'\mathbf{Q}\mathbf{u})\, dt \tag{4.18}$$

over an appropriate time interval, the limits usually being $t = 0$ to $t = \infty$.

To begin with, $\mathbf{P}$ and $\mathbf{Q}$ can be considered symmetric, since if they are not symmetric they can re-expressed as such.[4] From observation of these equations, it is also reasonable to assume a solution of the form $I_0 = \mathbf{r}'\mathbf{\Pi}\mathbf{r}$, where $\mathbf{\Pi}$ is symmetric. If the form of the optimal case is not symmetric then by uniqueness, it will not be a solution to the optimal programming equation. If this substituted into Eq. 4.16 for the optimal control

$$\frac{\partial}{\partial T}(\mathbf{r}'\mathbf{\Pi}\mathbf{r}) = \min_u \left\{ \mathbf{r}'\mathbf{P}\mathbf{r} + \mathbf{u}'\mathbf{Q}\mathbf{u} + \frac{\partial(\mathbf{r}'\mathbf{\Pi}\mathbf{r})}{\partial \mathbf{r}} \cdot (\mathbf{A}\mathbf{r} + \mathbf{B}\mathbf{u}) \right\} \tag{4.19}$$

$$\mathbf{r}'\frac{\partial \mathbf{\Pi}}{\partial T}\mathbf{r} = \min_u \{\mathbf{r}'\mathbf{P}\mathbf{r} + \mathbf{u}'\mathbf{Q}\mathbf{u} + 2\mathbf{r}'\mathbf{\Pi}(\mathbf{A}\mathbf{r} + \mathbf{B}\mathbf{u})\} \tag{4.20}$$

By taking the derivative with respect to $\mathbf{u}$ and setting to zero, the right hand side is minimized when

$$2\mathbf{u}'\mathbf{Q} + 2\mathbf{r}'\mathbf{\Pi}\mathbf{B} = 0 \tag{4.21}$$

[3] How do we know it is Markovian in nature? Here it is obvious that the rate of change only depends on the state of the system at that time. For the discrete case, the Markovian nature is evident from the lack of terms $\mathbf{r}(i-1)$ or earlier.

[4] Interestingly, this is not the case for a quantum cost function

or, noting that $\mathbf{\Pi}$ and $\mathbf{Q}$ are symmetric

$$\min_u \mathbf{u} = \mathbf{u}_0 = \mathbf{K}\mathbf{r} \tag{4.22}$$

where

$$\mathbf{K} = \mathbf{Q}^{-1}\mathbf{B}'\mathbf{\Pi} \tag{4.23}$$

This gives us the optimal control $\mathbf{u}_0$ in terms of a matrix $\mathbf{K}$ multiplied by the state $\mathbf{x}$.

Substituting the optimal control $\mathbf{u}_0 = -\mathbf{Q}^{-1}\mathbf{B}'\mathbf{\Pi}\mathbf{r}$ given by Eq. 4.22 back into the right hand side of Eq. 4.20 produces

$$\mathbf{r}'\frac{\partial \mathbf{\Pi}}{\partial T}\mathbf{r} = \mathbf{r}'\mathbf{P}\mathbf{r} + (\mathbf{Q}^{-1}\mathbf{B}'\mathbf{\Pi}\mathbf{r})'\mathbf{Q}(\mathbf{Q}^{-1}\mathbf{B}'\mathbf{\Pi}\mathbf{r}) + 2\mathbf{r}'\mathbf{\Pi}(\mathbf{A}\mathbf{r} - \mathbf{B}(\mathbf{Q}^{-1}\mathbf{B}'\mathbf{\Pi}\mathbf{r})) \tag{4.24}$$

$$\mathbf{r}'\frac{\partial \mathbf{\Pi}}{\partial T}\mathbf{r} = \mathbf{r}'\mathbf{P}\mathbf{r} + \mathbf{r}'\mathbf{\Pi}\mathbf{B}\mathbf{Q}^{-1}\mathbf{B}'\mathbf{\Pi}\mathbf{r} + 2\mathbf{r}'\mathbf{\Pi}\mathbf{A}\mathbf{r} - 2\mathbf{r}'\mathbf{\Pi}\mathbf{B}\mathbf{Q}^{-1}\mathbf{B}'\mathbf{\Pi}\mathbf{r} \tag{4.25}$$

$$\mathbf{r}'\frac{\partial \mathbf{\Pi}}{\partial T}\mathbf{r} = \mathbf{r}'(\mathbf{P} + 2\mathbf{\Pi}\mathbf{A} - \mathbf{\Pi}\mathbf{B}\mathbf{Q}^{-1}\mathbf{B}'\mathbf{\Pi})\mathbf{r} \tag{4.26}$$

By replacing $2\mathbf{r}'\mathbf{\Pi}\mathbf{A}\mathbf{r}$ with its symmetric counterpart $\mathbf{r}'(\mathbf{\Pi}\mathbf{A} + \mathbf{A}'\mathbf{\Pi})\mathbf{r}$ we arrive at the following differential equation for the symmetric matrix $\mathbf{\Pi}$

$$\frac{\partial \mathbf{\Pi}}{\partial T} = \mathbf{P} + \mathbf{\Pi}\mathbf{A} + \mathbf{A}'\mathbf{\Pi} - \mathbf{\Pi}\mathbf{B}\mathbf{Q}^{-1}\mathbf{B}'\mathbf{\Pi} \tag{4.27}$$

This belongs to a class of equations known as Riccati equations. The boundary condition is specified by considering the limit at $T \to 0$ where the performance criteria given by $I_0 = \mathbf{r}'\mathbf{\Pi}\mathbf{r}$ must equal zero, so that $\mathbf{\Pi}(0) = 0$. Thus the equations for the optimal control $\mathbf{u}_0$ and the optimal performance criteria I_0 have been developed, and the solution is complete.

Separation

It is pertinent to note the *separation* structure of the optimal control. It is a linear function of the state of the system $\mathbf{r}$, given by the control matrix $\mathbf{K}$. This matrix depends on the cost specified (via $\mathbf{Q}$), the way the system react to the control $\mathbf{B}$ and on the evolving matrix Π. The linear structure is important part of the solution, and the separation between the state vector and the control matrix makes the solution easy to implement.

Often the time over which the control is applied is not specified, and could be implemented for such a time that the control matrix $\mathbf{K}$, dependent on Π, has reached its steady state value. If the control designer assumes that the length of time the control is applied is infinite, then the steady state value is the appropriate control to apply, and they can solve the Ricatti equations for the steady state solution and

encode that into the control matrix.

$$0 = \mathbf{P} + \mathbf{\Pi A} + \mathbf{A'\Pi} - \mathbf{\Pi B Q}^{-1}\mathbf{B'\Pi} \tag{4.28}$$

This is termed the infinite horizon situation.

4.4 The LQG

The previous calculations have been a introduction to the sort of themes that are involved with optimization theory and LQ systems. However, the real power of control theory involves the ability to deal with not only imperfect measurement, but noise in both the system and the measurement signal. The paradigm example for this theory is the Kalman filter, which uses the gain matrix $\mathbf{K}$ defined previously (Eq. 4.23), and multiplies it by the *best estimate* $\tilde{\mathbf{x}}$ of the system state, will provides the optimum control policy. The proof that the same solution applies for both the perfectly understood system and one with noise and imperfect knowledge is provided by the certainty equivalence principle, which will be outlined below.

In this section calculations will be performed in continuous time.

4.4.1 A new system

Now a new model is introduced, that incorporates noise into the system and the measurement signal. This is done by the addition of a white noise vector ϵ, related to the Wiener increments by

$$\int \epsilon \, dt = dW$$

. The new model for the system evolution becomes

$$\dot{\mathbf{r}} = \mathbf{Ar} + \mathbf{Bu} + \epsilon_1 \tag{4.29}$$

where $\mathbf{r}$ is a vector describing the state, $\mathbf{u}$ is the vector describing the controls applied to the system, and $\mathbf{A}$ and $\mathbf{B}$ are matrices that describe how the system evolves as a linear function of $\mathbf{r}$ and $\mathbf{u}$. Obviously this system is still Markovian and evolves linearly with $\mathbf{r}$ and $\mathbf{u}$.

Furthermore, a measurement signal $\mathbf{y}$ shall be introduced, which provides some information about the state of the system. The information may be complete, or it may only partially observe the system, so the measurement signal will be a expressed as a linear function of the state vector $\mathbf{r}$. Most importantly, the measurement signal is corrupted by another white noise ϵ_2. It will have the form

$$\mathbf{y} = \mathbf{Cr} + \epsilon_2 \tag{4.30}$$

These two noise sources ϵ_1 and ϵ_2 may be correlated; in fact, by changing the angle between the LO and the system signal, from section 2.4, one can vary the amount of correlation. This manipulation of the system (quantum) noise and measurement noise will be explored in the next chapter; suffice to say, control theory is capable of dealing with any amount of correlation. The correlations can generally be expressed as

$$\text{cov}\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix} = \begin{bmatrix} \mathbf{N} & \mathbf{L} \\ \mathbf{L}' & \mathbf{M} \end{bmatrix}$$

For this section it will be assume for convenience that $\mathbf{L} = 0$.

4.4.2 Best estimate

The best estimate $\tilde{\mathbf{r}}$, whose dynamics are provided by the Kalman filter, is both a L*east* S*quares* E*stimate* (LSE) and a M*aximum* L*ikelyhood* E*stimate* (MLE). That is, it is an LSE in the sense that the expected difference between the real state of the system $\mathbf{r}$ and the best estimate $\tilde{\mathbf{r}}$, as well as the expected difference squared, has been minimized. The best estimate is also an MLE, well, because it is *most likely* state of the system.

This lends itself to the geometrical interpretation of the best estimate $\tilde{\mathbf{r}}$, being the projection of the actual state $\mathbf{x}$ onto the vector space of observed measurements, $\sum \mathbf{y}$. If the projection is given as the expectation of $\mathbf{r}$ given $\sum \mathbf{y}$

$$\text{E}\left[\mathbf{r}|\sum \mathbf{y}\right] = \tilde{\mathbf{r}}$$

then the difference $\tilde{\mathbf{r}} - \mathbf{x}$, orthogonal to $\sum \mathbf{y}$ is minimized.

Along with the best estimate $\tilde{\mathbf{r}}$, the uncertainty, or variance $\mathbf{V}$ of the best estimate is also available. This provides some indication of the accuracy of $\tilde{\mathbf{r}}$.

Another way to look at this is to say that given the history of measurement results, the distribution of the actual state will have a mean $\tilde{\mathbf{r}}$, with a variance $\mathbf{V}$. This distribution evolves as more measurements are taken, as system evolves as well.

The solution to the problem of best estimate is given by the celebrated Kalman filter, and for the situation above (Eq. 4.29 and Eq.4.30) it is updated with each measurement a very similar fashion to the system equations. For the best estimate,

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{V}\mathbf{C}'\mathbf{M}^{-1}(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}) \tag{4.31}$$

and the variances evolve as

$$\dot{\mathbf{V}} = \mathbf{N} + \mathbf{A}\mathbf{V} + \mathbf{V}\mathbf{A}' - \mathbf{V}\mathbf{C}'\mathbf{M}^{-1}\mathbf{C}\mathbf{V} \tag{4.32}$$

where $\mathbf{N}$ and $\mathbf{M}$ are the covariances for the two noise sources ϵ_1 and ϵ_2. Derivations of these relations abound; some particularly good developments are in [14] [15].

4.4.3 Some old ideas renewed

In this subsection, some previous concepts are briefly re-visited; the performance criteria and the Principle of Optimality. With the introduction of noise, these ideas take on new forms. Using these new forms, one can arrive at the solution to the LQG problem, being the Kalman filter. However, instead of covering that development, it is easier to employ a concept by Whittle [15] which hands the solution over in a very simple way.

The performance revisited

Now that a new system has been defined, by introducing a stochastic element to the mathematics, it is necessary to redefine the performance criteria to take this into account. It is not appropriate to measure the performance of a control policy using the previous performance I, over a single run of the system. Perhaps that run will have statistically larger deviations than normal, implying worse performance than what might be usual. The solution is to use the expectation operator $\mathrm{E}[\,]$ to find the expected, or average, cost over the time that the control policy is implemented, denoted J. Thus for stochastic systems, the performance is given as the *expected* total cost, and is evaluated as

$$J = \mathrm{E}[I] = \mathrm{E}\left[\sum_t c(\mathbf{r}_t, \mathbf{u}_t, t)\right] \tag{4.33}$$

$$J = \mathrm{E}\left[\int c\left(\mathbf{r}(t), \mathbf{u}(t), t\right)\, dt\right] \tag{4.34}$$

The Principle of Optimality revisited

Now, for interest, reconsider the Principle of Optimal control once again. The optimal control should be the one that ensures the smallest (infimal) expected cost over the time of implementation. Similarly if the expected cost is minimized over the set of *all* control policies, denoted by π, then this will be the smallest expected cost, denoted[5] by $J_0(\mathbf{r})$ and can be mathematically expressed as

$$J_0(\mathbf{r}(t)) = \inf_{\pi} \mathrm{E}_{\pi}\left[\int_0^h c(\mathbf{r}, \mathbf{u}) dt\right] \tag{4.35}$$

[5]Here the subscript 0 implies that it is the best performance, being numerically the smallest. It results from the optimal control.

$$J_0(\mathbf{r}_0) = \inf_{\pi} \mathrm{E}_{\pi} \left[\sum_{t=0}^{h} c(\mathbf{r}_t, \mathbf{u}_t, t) \right] \tag{4.36}$$

Now a restatement of the Principle of Optimality expressed the fact that the infimal expected cost $J_0(\mathbf{r}_t)$ at any time t obeys the recursive equation

$$J_0(\mathbf{r}_t) = \inf_{u_t} \left(c(\mathbf{r}_t, \mathbf{u}_t, t) + \mathrm{E}\left[J_0(\mathbf{r}_{t+1}) | \mathbf{r}_t, \mathbf{u}_t \right] \right) \tag{4.37}$$

The second term in the brackets $\mathrm{E}\left[J_0(\mathbf{r}_{t+1}) | \mathbf{r}_t, \mathbf{u}_t \right]$ is the expectation of $J_0(\mathbf{r}_{t+1})$ at time $t+1$, conditional on, or given that, the state and control at time t is $\mathbf{r}_t$ and $\mathbf{u}_t$. Notice that now the infimal process is taken over the control $\mathbf{u}_t$, instead of over the whole control process. This recursive nature lends itself to an easy solution.

With some thought, it can be seen that this is equivalent to the previous statement of the Optimality Principle; that an optimal control sequence (read optimal infimal expected cost) has the property that if it passes through a particular state, then remaining controls also constitute an optimal sequence. However, instead of using this, it is easier to appeal to the Certainty-Equivalence Principle, given in the next.

4.4.4 Certainty-equivalence principle

The Certainty-Equivalence Principle (CEP) [15] is a powerful notion that allows one to replace the state in the control loop with the best state estimate. The CEP tells us that the appropriate control for a system with noise and imperfect measurement introduced, is the same form as that for without noise and perfect measurement, namely

$$\mathbf{u}_0 = \mathbf{K}\tilde{\mathbf{r}} \tag{4.38}$$

(See Eq.4.22) except one has replaced the state of the system (available under perfect observation) with the best estimate for the state of the system $\tilde{\mathbf{r}}$, which comes from applying the Kalman filter to the measurement signal. This is the optimal control solution for the LQG problem.

It is obvious that this solution has the same separation structure as for the LQR solution (See section 4.3.2). Furthermore, the control matrix $\mathbf{K}$ does not depend on the measurement history, and assuming the end of the control process is at a time $t = \infty$ (infinite horizon), then it will assume a steady state form, and the total control only depends linearly on the best state estimate $\tilde{\mathbf{r}}$.

4.5 Conclusion

The desire of the control designer has been quantified in this chapter with the introduction of the cost function. Next, a Principle of Optimality is utilized to find

the 'optimal' control to apply to a system, that minimizes the cost function. Two model have been introduced in this chapter, the LQR and the LQG. The LQG model is a generalized version of the LQR, with stochastic noise being introduced into the system and the measurement process. The Certainty-Equivalence Principle states that the two have the same control process, but the Kalman filter is employed to determine the best state estimate.

Chapter 5

Quantum Control Theory

Now that a basis of control theory has been laid down, it is time to apply this to the quantum systems previously considered. This work was covered by Doherty and Jacobs [10], and more rigorously derived by Doherty, Tan, Parkins and Walls [16], To begin with, an appropriate 'quantum' cost function is proposed and discussed, as well as some brief ideas of feedback mechanisms. Finally, the optimal controls are presented

5.1 A good quantum cost

A discussed earlier in section 4.1, considerable importance lies upon the selection of an appropriate cost function. In the design of a cost function for a quantum system, Doherty and Jacobs [10] suggested a quantum cost function of the form

$$J_q = \int dt \, \mathrm{Tr}\left(\mathbf{r}'\mathbf{P}\mathbf{r}\rho_c\right) + \langle \mathbf{u}'\mathbf{Q}\mathbf{u}\rangle_c \tag{5.1}$$

where $\langle\,\rangle_c$ indicates an average of the classical random variables, namely the controls $\mathbf{u}$, and $\mathbf{r}$ is the vector of *operators*

$$\mathbf{r} = \begin{pmatrix} x \\ p \end{pmatrix} \tag{5.2}$$

This satisfies all of the criteria referred to earlier in section 4.1, most notably it is quadratic in both position and momentum. Furthermore it incorporates the best knowledge of the system state by taking the trace of the density operator. Here the density operator takes on two subtly distinct roles; first as an operator describing the quantum system, and secondly as a function containing the best estimate of the system state, to our knowledge.

Expanding the quantum state cost

Now expand the 1st term in the quantum cost equation 5.1

$$\begin{aligned} \mathrm{Tr}\left(\mathbf{r}'\mathbf{P}\mathbf{r}\rho_c\right) &= \mathrm{Tr}\left(\begin{pmatrix} x & p \end{pmatrix}\mathbf{P}\begin{pmatrix} x \\ p \end{pmatrix}\rho_c\right) && (5.3)\\ &= \mathrm{Tr}\left(P_{11}x^2\rho_c + P_{12}xp\rho_c + P_{21}px\rho_c + P_{22}p^2\rho_c\right) && (5.4)\\ &= P_{11}\langle x^2\rangle + P_{12}\langle xp\rangle + P_{21}\langle px\rangle + P_{22}\langle p^2\rangle && (5.5)\end{aligned}$$

If we assume that the matrix $\mathbf{P}$ is symmetric, then this reduces to

$$\begin{aligned} \mathrm{Tr}\left(\mathbf{r}'\mathbf{P}\mathbf{r}\rho_c\right) &= P_{11}\langle x^2\rangle + P_{12}(\langle xp\rangle + \langle px\rangle) + P_{22}\langle p^2\rangle && (5.6)\\ &= P_{11}(\langle x^2\rangle - \langle x\rangle^2 + \langle x\rangle^2)\\ &\quad +P_{12}(\langle xp\rangle + \langle px\rangle - \langle x\rangle\langle p\rangle + \langle x\rangle\langle p\rangle)\\ &\quad +P_{22}(\langle p^2\rangle - \langle p\rangle^2 + \langle p\rangle^2) && (5.7)\\ &= P_{11}(V_x + \langle x\rangle^2) + P_{12}(2C + \langle x\rangle\langle p\rangle)\\ &\quad +P_{22}(V_p + \langle p\rangle^2) && (5.8)\end{aligned}$$

where V_x and V_p are the variances of x and p respectively, and C is the symmetric covariance of x and p given by

$$C = \tfrac{1}{2}\langle xp + px\rangle - \langle x\rangle\langle p\rangle \tag{5.9}$$

leaving us with

$$\begin{aligned} \mathrm{Tr}\left(\mathbf{r}'\mathbf{P}\mathbf{r}\rho_c\right) &= P_{11}V_x + P_{12}2C + P_{22}V_p + P_{11}\langle x\rangle^2 + P_{12}\langle x\rangle\langle p\rangle\\ &\quad +P_{22}\langle p\rangle^2 && (5.10)\\ &= \langle\mathbf{r}\rangle'\mathbf{P}\langle\mathbf{r}\rangle + \mathrm{Tr}\left(\mathbf{P}\mathbf{V}\right)) && (5.11)\end{aligned}$$

where V is the covariance matrix given by

$$\mathbf{V} = \begin{pmatrix} V_x & C \\ C & V_p \end{pmatrix} \tag{5.12}$$

Thus,

$$J_q = \int dt\, \langle\mathbf{r}\rangle'\mathbf{P}\langle\mathbf{r}\rangle + \mathrm{Tr}\left(\mathbf{P}\mathbf{V}\right)) + \langle\mathbf{u}'\mathbf{Q}\mathbf{u}\rangle_c \tag{5.13}$$

5.1.1 Selecting a cost function

Now that a general form of J_q is available, is it necessary to select the the terms of the matrix $\mathbf{P}$. It is an interesting question to ask how quantum mechanics restricts

the form of $\mathbf{P}$, since it won't be possible to reduce the uncertainties of position and momentum below that dictated by the Heisenberg uncertainty principle. Yet it is the total cost to be minimized, so that it is only the size of the elements of $\mathbf{P}$ relative to each other, that are important.

Considering that the ambition of most atom-optic laboratories is the cooling and confinement of an atom, one obviously desirable goal would be to place the harmonically confined atom in the ground state of the potential. This is the minimum energy eigenstate, and is a gaussian wave function completely defined by

$$\langle x \rangle = 0 \tag{5.14}$$
$$\langle p \rangle = 0 \tag{5.15}$$
$$\langle x \rangle^2 = V_x = \frac{\hbar}{2m\omega} \tag{5.16}$$
$$\langle p \rangle^2 = V_p = \tfrac{1}{2} m\hbar\omega \tag{5.17}$$

The choice of

$$\mathbf{P} = \begin{pmatrix} \frac{1}{2}m\omega^2 & 0 \\ 0 & \frac{1}{2m} \end{pmatrix} \tag{5.18}$$

will provide a quantum cost function of

$$J_q = \int dt\ \langle \mathbf{r} \rangle' \mathbf{P} \langle \mathbf{r} \rangle + \mathrm{Tr}\left(\mathbf{PV}\right)) + \langle \mathbf{u}'\mathbf{Q}\mathbf{u} \rangle_c \tag{5.19}$$
$$= \int dt\ \frac{1}{2}m\omega^2 \langle x \rangle^2 + \frac{\langle p \rangle^2}{2m} + \frac{1}{2}m\omega^2 V_x + \frac{V_p}{2m} + \langle \mathbf{u}'\mathbf{Q}\mathbf{u} \rangle_c \tag{5.20}$$

Here we can see that this form of cost will try to reduce both the deviations $\langle x \rangle$ and $\langle p \rangle$ from the origin, as well as the variances V_x and V_p, around the origin. Notice also that the choice of $\mathbf{P}$ has placed all of the terms in units of energy.

It is also required to determine the form of $\mathbf{Q}$, which will be the cost incurred with the application of the control. Although a lot of consideration was taken to include this term (see section 4.1.1), this is not a common consideration for most physicist, so a relatively simple form can be assumed [10],

$$\mathbf{Q} = q^2\mathbf{P} = q^2 \begin{pmatrix} \frac{1}{2}m\omega^2 & 0 \\ 0 & \frac{1}{2m} \end{pmatrix} \tag{5.21}$$

In the limit of small q, the cost function becomes a measure of localization in phase space, independent of control strength; a cost most physicist would appreciate.

5.2 The quantum control model

The equations for the dynamics of the quantum system are

$$d\langle x\rangle = \frac{\langle p\rangle}{m}dt + 2\sqrt{2\eta k}\,V_x\,dW \tag{5.22}$$

$$d\langle p\rangle = -m\omega^2\langle x\rangle dt + 2\sqrt{2\eta k}\,C\,dW \tag{5.23}$$

which can be expressed in a form more suitable for control theory as

$$\dot{\langle x\rangle} = \frac{\langle p\rangle}{m} + 2\sqrt{2\eta k}\,V_x\,\epsilon \tag{5.24}$$

$$\dot{\langle p\rangle} = -m\omega^2\langle x\rangle + 2\sqrt{2\eta k}\,C\,\epsilon \tag{5.25}$$

so that

$$d\mathbf{r} = \mathbf{A}\mathbf{r}\,dt + \mathbf{B}\mathbf{u} + \epsilon_1 \tag{5.26}$$

where

$$\mathbf{r} = \begin{pmatrix} x \\ p \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} 0 & 1/m \\ -m\omega^2 & 0 \end{pmatrix} \tag{5.27}$$

and $\mathbf{Bu}$ is a control matrix and control signal, yet to be specified, and ϵ_1 is the white noise vector with

$$\mathrm{E}[\epsilon_1] = 0$$

$$\mathrm{cov}[\epsilon_1] = \begin{pmatrix} 8k\eta V_x^2 \\ 8k\eta C^2 \end{pmatrix}$$

5.3 The feedback mechanism

There exists a relationship between the best estimated state of a system, and the nature of the density operator. Both represent the knowledge of an observer has on the state of a system in a statistical form. Thus the mapping of the expectation of x and p and the best estimate of x and p

$$\begin{bmatrix} \mathrm{Tr}[x\rho_c] \\ \mathrm{Tr}[p\rho_c] \end{bmatrix} \mapsto \tilde{\mathbf{r}}$$

provides a guide to implementing the feedback mechanism.

Only if the feedback terms affect the dynamics of the quantum system linearly can the results of the previous chapter on control theory be applied. Thus we are

confined to a Hamiltonian of the form

$$H_{fb} = f(\,\mathrm{Tr}[x\rho_c], \mathrm{Tr}[p\rho_c]\,)x \,+\, g(\,\mathrm{Tr}[x\rho_c], \mathrm{Tr}[p\rho_c]\,)p$$

where f and g are functions yet to be determined.

Drawing on the results from the previous chapter, it can be seen that the form of function f and g are linear in $\mathrm{Tr}[x\rho_c]$ and $\mathrm{Tr}[p\rho_c]$, as the optimal control $\mathbf{u}_0$ is a linear function of the best estimate $\tilde{\mathbf{r}}$, given by $\mathbf{K}$.

It is now appropriate to determine the form of $\mathbf{K}$. We have

$$\mathbf{K} = \mathbf{Q}^{-1}\mathbf{B}'\mathbf{\Pi} \tag{5.28}$$

where $\mathbf{\Pi}$ is given by the solution to the steady-state Ricatti equation (See Eq.4.28)

$$0 = \mathbf{P} + \mathbf{\Pi}\mathbf{A} + \mathbf{A}'\mathbf{\Pi} - \mathbf{\Pi}\mathbf{B}\mathbf{Q}^{-1}\mathbf{B}'\mathbf{\Pi} \tag{5.29}$$

where $\mathbf{A}$ is given by Eq. 5.27, $\mathbf{P}$ by Eq.5.18 and $\mathbf{Q}$ by Eq.5.21. A form of the feedback needs to be assumed here; another assumption for simplicity will bode well. If

$$\mathbf{B} = \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{5.30}$$

so two individual elements of control are fed into both the position and momentum, then the specification for $\mathbf{K}$ is complete.

Solving the Ricatti equation leads to a elementary control matrix

$$\mathbf{K} = 1/q\mathbf{I} \tag{5.31}$$

This can be easily interpreted. As $q \to 0$, the cost to apply control becomes unimportant, and the strength of the feedback becomes very high, driving the system to the ground state. Furthermore, the feedback into the $\langle x \rangle$ term is only a function of $\mathrm{Tr}[x\rho_c]$ and likewise for the p coordinate; though this lies somewhat on the judicious choice of $\mathbf{B}$.

Thus the form of the feedback Hamiltonian is given by

$$\begin{aligned} H_{fb} &= \mathbf{K}\tilde{\mathbf{r}} \\ &= f(\,\mathrm{Tr}[x\rho_c]\,)x \,+\, g(\mathrm{Tr}[p\rho_c]\,)p \\ &= 1/q\mathrm{Tr}[x\rho_c]\,x \,+\, 1/q\mathrm{Tr}[p\rho_c]\,p \end{aligned}$$

5.4 The end result

The resulting dynamical system of equations for $\langle x \rangle$ and $\langle p \rangle$ and their associated covariances have been explored by Doherty and Jacobs [10], and need not be repeated

here. Suffice to say that localization phase space is achieved, to a degree that depends on many of the parameters previously discussed. Most notably, the atom is confined in a way that is roughly linear with the cost of the control q, and inversely with the detector efficiency η and the measurement strength k (See Eq.3.14).

Finally, these results are compared in the paper by Doherty and Jacobs [10] with the model of direct feedback developed by Milburn and Wiseman [17]. It can be seen that although noise can be completely 'eliminated' from the measured system, the system is forced to exponentially grow in momentum space, due to back action. No Hamiltonian, linear in x and p, and directly proportional to the measurement signal, can damp this growth in momentum space. Thus direct feedback does not offer the advantages that state-estimation does.

Chapter 6

Robust Control Theory

Although the Kalman filter is the optimal control process to apply to a system where noise is introduced in a prescribed way, an infinite number of other control process exists, and it can be shown that other forms of filters will provide better control under differing circumstances.

The search for the risk-sensitive filter arose out of the desire for a filter, similar to the Kalman filter, but with the property of "good performance under any nominal conditions and acceptable performance for signal and noise conditions other than the nominal which can range over the whole allowable classes of possible characteristics." [18]

This chapter briefly covers some of the ideas surrounding the relatively new field of robust control. A redefined exponentiated quadratic cost function will relate to a new control policy; a risk-sensitive modification on the Kalman filter that provides an element of robustness to the control system. This new control filter is often termed a L*inear* E*xponential* Q*uadratic* G*aussian* (LEQG), as the quadratic cost function has now been exponentiated.

The development of this LEQG theory and the associated solutions are highly mathematical, and a rigorous presentation is provided by Peter Whittle [15]. This chapter contains only a brief overview.

6.1 A new cost, and a new control

A natural question to ask would be "What would happen if one changed the cost function $c(\mathbf{r}, \mathbf{u})$?" Obviously, reworking the above calculations would most often result in a different control policy. Being different to the Kalman filter solution, is this control still optimal? The answer to this is yes, but the resulting control policy is optimal to minimize a different cost function.

A new cost

A new cost was suggested by Jacobs in 1973, which was very closely related to the original cost

$$c_{rs}(\mathbf{r}, \mathbf{u}) \quad = \quad e^{\mu \int \mathbf{r'Pr+u'Qu}\ dt} \tag{6.1}$$

This is merely a re-scaling of the original quadratic cost function outlined in the LQG problem by a parameter μ, and exponentiated. This new parameter μ is often called the *risk-sensitive parameter*, and the control designer is free to set this, within the bounds of maintaining the integrity of the mathematical structure of the model. This new formulation of the cost lends itself to the description of an L*inear* E*xponential* Q*uadratic* G*aussian* (LEQG) problem, where the cost is now an exponentiated quadratic.

Reduction to the Kalman filter

One interesting property of this new cost is that the free parameter μ allows the control designer to reduce the formulation to that of the LQG problem. More precisely, in the limit $\mu \to 0$, the control policy that results from the optimization process reproduces the Kalman filter solution to the LQG problem.

This can be seen by expanding the exponential for small μ

$$\begin{aligned}
J_{LEQG} &= \min_{\mathbf{u}} \mathrm{E}\left[c_{rs}(\mathbf{r}, \mathbf{u})\right] \\
&= \min_{\mathbf{u}} \mathrm{E}\left[e^{\mu \int \mathbf{r'Pr+u'Qu}\ dt}\right] \\
&\simeq \min_{\mathbf{u}} \mathrm{E}\left[1 + \mu \int \mathbf{r'Pr} + \mathbf{u'Qu}\ dt\right] \\
&= 1 + \mu \min_{\mathbf{u}} \mathrm{E}\left[\int \mathbf{r'Pr} + \mathbf{u'Qu}\ dt\right] \\
&= 1 + \mu \min_{\mathbf{u}} \mathrm{E}\left[c(\mathbf{r}, \mathbf{u})\right]
\end{aligned}$$

where $c(\mathbf{r}, \mathbf{u})$ is obviously the quadratic cost from the original LQG formulation. So, the minimization of the exponentiated quadratic cost over the control policy *reduces* to a minimization of just the quadratic cost over the control policy as $\mu \to 0$, with the result being the same control policy of the Kalman filter.

Interpretation of the cost function

A substantial amount of literature discusses the interpretation of the risk-sensitive cost function. One easily accessible view is that by exponentiating the original

quadratic cost, one is penalizing deviations from the desired state *exponentially greater* than for the original LQG system. As such, the resulting control will tighter confine the system to the desired state with respect to large scale deviations, yet may suffer more deviations on a smaller scale.

Since μ is called the risk-sensitive parameter, then one can interpret $\mu \to 0$ as the *risk-neutral* case, which corresponds to the LQG system. Thus one can consider the risk-neutral, or LQG problem, as a special case of the larger risk-sensitive, or LEQG problem. This is the treatment given by Whittle [15], who also reflects on μ being a measure of 'optimism' or 'pessimism' of the control designer, as to whether events will turn out in their favour or not. The risk-sensitive parameter can be considered an experimental parameter; a knob that the experimentalist can turn up to tighter confine the perturbations of a system that are not modelled by the feedback mechanism.

6.2 Perspective

Consider two quotes which highlights the utility of the LEQG formalism. First a paper on quantum feedback, by Doherty, Habib, Jacobs, Mabuchi and Tan [19],

> If parameters of the model of the system are not in fact well known then the control that is optimal for the nominal model may in fact be a very poor control loop for models with similar but not identical values of the parameters. This problem can be particularly pronounced in systems with large number of degrees of freedom and the solution of this problem is the domain of robust control

In a paper by Boel, James and Petersen the essence of robustness is expounded in a few short words [20]

> [Robustness / LEQG] provides good performance under nominal conditions and acceptable otherwise.

Although robustness is cited as one of the greatest benefits of an LEQG implementation, perhaps it should be considered that the LEQG formalism is just as easy to implement, as a dynamical programming code for a digital signal processor for example, as an LQG filter. Furthermore it contains the LQG filter as a special case in its operation ($\mu \to 0$). Finally, it allows an experimentalist to control the degree of 'aggressiveness' with which the filter acts.

Chapter 7

Robust Quantum Control Theory

The theory behind the application of risk-sensitive control of a quantum system is relatively new , and although a few papers have considered this situation [21] [22], there remains some deeper elements to this merger of theories that needs greater consideration.

This chapter details some of the simulation results achieved for the LEQG filter applied to the quantum system of an atom being observed, with feedback.

7.1 The simulation

The program used to simulate the quantum control problem was written in MATLAB, using the explicit Euler method. The implementation of the filter was taken in the steady state (See section 4.3.2) as well as the variances, which also take on a steady state form, invariant to the measurement signal.

7.1.1 Separating the nominal and physical models

The program code is set out in Appendix, chapter 9, and is written to calculate the performance of a control when the nominal model (that which is being used by the control system) differs from the physical model (the system which the control system is applied to).

```
No_delta = 11;   x_start = 0;    x_finish = 1
x = linspace(x_start,x_finish,No_delta);
mu = 0.2
MSE_LQG = zeros(5,No_delta);
MSE_LEQG = zeros(5,No_delta);
for d = 1:No_delta
      del = x(d);
      phys_init;
      runLQG;
      MSE_LQG(:,d) =  [mean(r(1,:).^2); mean(r(2,:).^2);
```

```
            mean(r(3,:).^2); mean(r(4,:).^2); mean(r(5,:))];
      runLEQG;
      MSE_LEQG(:,d) = [mean(r(1,:).^2); mean(r(2,:).^2);
            mean(r(3,:).^2); mean(r(4,:).^2); mean(r(5,:))];
end
save mu_0.2_del_-1-1.2_3.mat
```

To measure this, a parameter `x` is introduced into the program, running from `x_start` to `x_finish` with `No_delta` equally spaced points. A loop over the variable `d` selects these various `x`s and incorporates them into the physical model via the subroutine `phys_init`, and the physical model deviates from the nominal (filter) model. For example, to create a difference in between the physical and nominal model in the element A_{11} for the state evolution

$$\dot{\mathbf{r}} = \mathbf{A}\mathbf{r} + \mathbf{B}\mathbf{u} + \epsilon_1$$

the `phys_init` routine has the line

```
A_p = [del ,1/mass ;-mass*w^2,0];
```

Statistics of each run, in this case the mean-squared position, momentum, controls and mean performance, are logged in the matrices `MSE_LQG` and `MSE_LEQG`.

7.1.2 The different filters

The simulation subroutines `runLQG` and `runLEQG` simulate the same physical experiment with the different LQG and LEQG filters, and complete data for each run is stored in the matrix `r`, namely the position and momentum of the atom at each point, along with the control strength applied and the performance up to that point. Inside, the structure of `runLQG` and `runLEQG` are almost identical, with a loop over `i` calculating the state of the system for all `No_points` points. The subroutine `runLQG` looks like this below.

```
for i = 1 : No_points-1
  dW = randn*sqrt(dt);
  % Change in physical model and measure
  dr = physical( r(1:2,i), A_p,B_p, Y_p, H_p, N_p,u,dt,dW,del);
  dy = measure( r(1:2,i), H_p, N_c, dt, dW );
  % Change in filter and control
  dr_c = filterLQG(r_c, A_c, B_c, u, dy, Y_c, H_c, N_c, dt );
  r_c = r_c + dr_c;
  u = K*r_c;
  % records physical system, control and cost
```

```
  r(1:2,i+1) = r(1:2,i) + dr;
  r(3:4,i) = u;
  r(5,i+1) = L(r(1:2,i), Y_p, u, P, Q);
end
```

Some of the notable points in this loop are

- **dW :** The variable `dW`, equivalent to the typical Wiener increment, is defined using the generator `rand` built in to MATLAB, which provides a normally distributed random number with mean 0 and variance. By scaling with the square root of the time increment $\sqrt{dt}$, this variable is assured to satisfy the Itó calculus relation
$$\mathrm{E}\left[dW^2\right] = dt \tag{7.1}$$

- **subroutine 'physical' :** The physical model is evolved over the time dt using the subroutine `physical`. Starting with the initial state `r`, the random variable dW and the parameters for the physical system `A_p`, `B_p`, `Y_p`, `H_p`, `N_p` and the control vector `u` calculated from the previous state, the change in the system state `dr` is determined with

  ```
  function dr = physical( r, A_p,B_p,Y_p,H_p,N_p,u,dt,dW ,del)
  % Models the physical system
  dr = dt*(A_p*r+[0;del] + B_p*u)+Y_p*H_p'*dW*N_p^(-0.5);
  ```

- **subroutine 'measure' :** A measurement signal from the system is calculated via the subroutine `measure`,

  ```
  function dy = measure( r, H_p, N_c, dt, dW )
  % models the change in measurement
   dy = H_p*r*dt + sqrt(N_c)*dW;
  ```

- **subroutine 'filterLQG' :** Here is where the difference between `runLQG` and `runLEQG` is evident; the system is going to be subjected to two different filters, the LEQG and LQG. Thus for the `runLEQG` run, the filter `filterLEQG` is called. These subroutines calculates the change in the best estimate of the system `r_c`, using the filter parameters (different to those parameters of the physical model) and the codes

  ```
  function dr_c = filterLQG(r_c,A_c,
                      B_c,u,dy,Y_c,H_c,N_c,dt);
  % models changed in estimated state r_c for an LQG filter
  dr_c = dt*(A_c*r_c+B_c*u)
        +Y_c*H_c'*(dy-H_c*r_c*dt)*N_c^-1;
  ```

and

```
 function dr_c = filterLEQG(r_c, A_c,
                   B_c, u, dy, Y_c, H_c, N_c, mu, P, dt );
% models changed in estimated state r_c for an LQG filter
dr_c = dt*((A_c + mu*Y_c*P)*r_c + B_c*u)
      + Y_c*H_c'*(dy - H_c*r_c*dt)*N_c^-1;
```

7.1.3 Calculating the performance

Performance is calculated by the subroutine `L` as

```
function L = L(r, Y_p, u, P, Q)
% calculates the change in the cost function
L = (r'*P*r + u'*Q*u + trace(P*Y_p));
```

7.1.4 Displaying the results

Using a program `pplot`, graphs displaying the performance over `x` are shown; an example of such a plot is given by Fig.7.1. Here the performance of the risk-sensitive

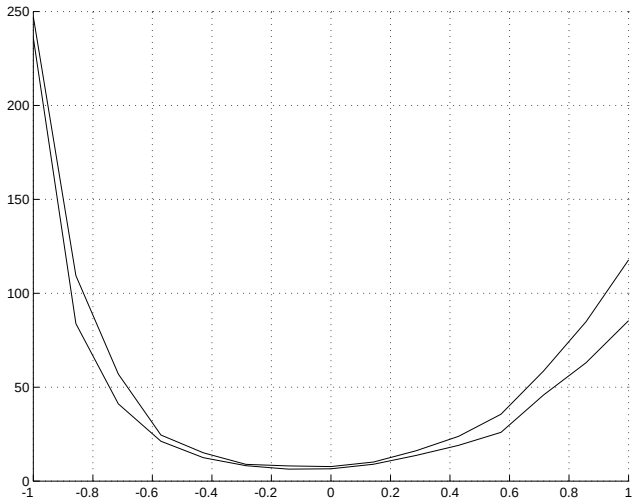

Figure 7.1: **Example of "pplot" output, displaying performance**

and risk-neutral filters on a quantum system are shown. The difference between the nominal and physical models is usually express as a parameter labelled as Δ along the x-axis The situation of no difference between the physical and nominal models is given by $\Delta = 0$, usually at the centre of the x-axis. Along the y-axis the performance

criteria is displayed, being the quadratic sum of the control and the deviations from the origin in phase space. Note that better performance implies a lower performance criteria.

7.1.5 Cost and performance

In measuring the performance of the LEQG and LQG filters, is was important to have a criteria that was consistent between both. It must be remembered that the LEQG filter is still an optimal control policy, yet for a different cost function, namely one that is exponentiated.

Here the performance is measured by the quadratic cost function; the point of robustness for the LEQG is to perform better for the same cost by the LQG, when there exists perturbations between the physical and nominal model.

Note that for no perturbation between the two models, the LQG filter will always out-perform the LEQG filter. This is because the LQG filter is the *optimal* filter for a performance criteria measured as a quadratic cost. Only when the models deviate does the LEQG have a chance of performing better.

7.2 Code check

It is important to check code against some simple situations where the solution is well understood. By removing the control, it was possible to model the random walk in momentum space.

7.2.1 Free run

One good check on the computer simulation is to check a sample path without any control. Since physically the system is still being perturbed by the measurement apparatus, the atom will undergo a random walk in momentum space, and this will be seen as a increase in both the average position and momentum for any particular run. Note that the average momentum $\mathrm{E}[\langle p \rangle]$ is still zero; however, in the one dimensional case considered in this thesis, it is the momentum squared $\mathrm{E}[\langle p^2 \rangle]$ related to the variance V_p, that is important. Form the graph one can see the variance growing, as with a random walk.

7.3 Results

Once the program was set up, it was then up to finding a physical model that *differs* from the one used in the control system, and seeing if the LEQG would out perform the LQG system.

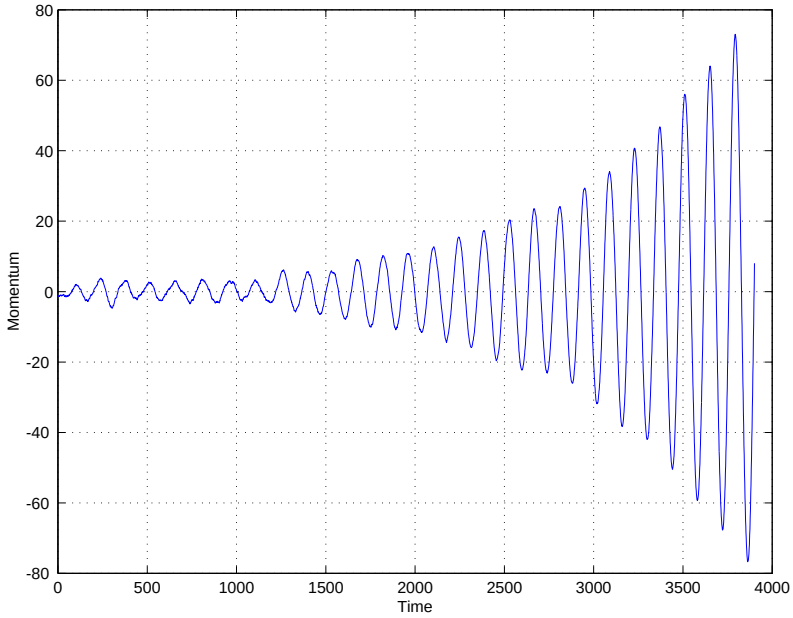

Figure 7.2: Random walk in momentum space

7.3.1 Zeeman

One possible perturbation to the system was that of a Zeeman shift. This implies the addition of a linear potential to the harmonic confinement of the atom. The addition of a linear term to the quadratic potential results in the shift of the origin (or the minimum). If one looks at the Hamiltonian for the mechanical motion

$$\begin{aligned} H &= \frac{p^2}{2m} + V(x) \\ &= \frac{p^2}{2m} + \tfrac{1}{2}m\omega^2 x^2 + Zeeman\ term \\ &= \frac{p^2}{2m} + \tfrac{1}{2}m\omega^2 x^2 + z_0 x \\ &= \frac{p^2}{2m} + \tfrac{1}{2}m\omega^2 \left(x + \frac{z_0}{m\omega^2}\right)^2 - \frac{z_0^2}{2m\omega^2} \end{aligned}$$

Notice that this changes the force applied to the particle, and this arises in the dynamical equations of the system as an extra constant term for $\dot{p}$ like

$$\begin{aligned} \dot{p} &= Force \\ &= -\frac{\partial}{\partial x}V(x) \end{aligned}$$

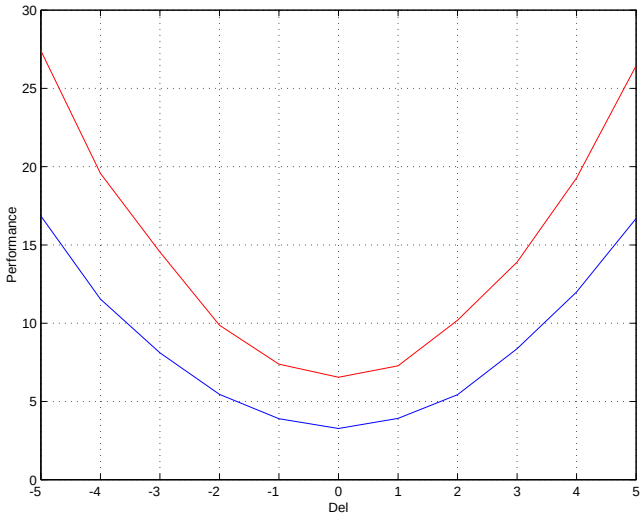

Figure 7.3: The performances of the LEQG and LQG filters under a Zeeman shift. The top line is the LEQG cost; the lower line is the LQG cost

$$= \quad -m\omega^2 x - z_0$$

and so the equations for the dynamical motion of the system will be changed like

$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & 1/m \\ -m\omega^2 & 0 \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} + \begin{bmatrix} 0 \\ -z_0 \end{bmatrix}$$

$$\rightarrow [A_p] \begin{bmatrix} x \\ p \end{bmatrix} + \begin{bmatrix} 0 \\ -\Delta \end{bmatrix}$$

where $[A_p]$ is the original matrix describing the system dynamics, and Δ being the perturbation parameter.

This was incorporated into the code for `physical.m` as below

```
function dr = physical(r,A_p,B_p,Y_p,H_p,N_p,u,dt,dW,del)
% Models the physical system
dr = dt*(A_p*r + [0;del] + B_p*u) + Y_p*H_p'*dW*N_p^(-0.5);
```

From Fig. 7.3 it can be seen that the LEQG filter does not handle the system as well as the LQG, even as the disturbance Δ increases.

Explanation

An explanation of this can be understood by realizing that a shift in potential means that both the LQG and LEQG controllers will be regulating the system to a point other than the origin. This means that both will suffer the same additional cost due to the atom rolling down the potential, and requiring a constant control to place it back. With both filters suffering the same cost displacement as a result of the Zeeman shift, the LQG will continue to out perform the LEQG for all displacements.

7.3.2 Antifriction

To take on a reverse-engineering approach, one can run a simulation where an arbitrary perturbation is placed between the physical and nominal model. If the simulation is successful, in the sense that the LEQG performs better than the LQG filter, a physically realistic mechanism to produce that perturbation would need to be derived. This is the situation for the addition of a term to the system dynamics that mimics a frictional or antifrictional term. Although the LEQG filter outperforms the LQG filter, no physical mechanism for this has been spelt out.

The perturbation

This was almost achieved for a perturbation to the dynamics matrix for the physical model `A_p`. This perturbation added a term that coupled the position to the rate of change of the position

$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} \Delta & 1/m \\ -m\omega^2 & 0 \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix}$$

$$= [A_p(\Delta)] \begin{bmatrix} x \\ p \end{bmatrix}$$

This system is equivalent to the two couple differential equations

$$\dot{x} = \frac{p}{m} + \Delta x$$

$$\dot{p} = -m\omega^2 x$$

which can be solved as

$$\ddot{x} = -\omega^2 x + \Delta \dot{x}$$

This is the equation for a damped harmonic oscillator; with the term $\Delta\dot{x}$ being proportional to the velocity, energy is being taken into or out of the system. The term 'anti-friction' is a misnomer; if the sign of Δ is negative, then energy is being removed from the system, and the amplitude of the oscillations is being damped via

the 'frictional' term Δx. For $\Delta < 0$ energy is being introduced into the system, and this is being described as 'antifriction'.

This was encoded into the simulation by changing the matrix `A_p` in the initialization for the physical model, via

```
A_p = [del ,1/mass ;-mass*w^2,0];
```

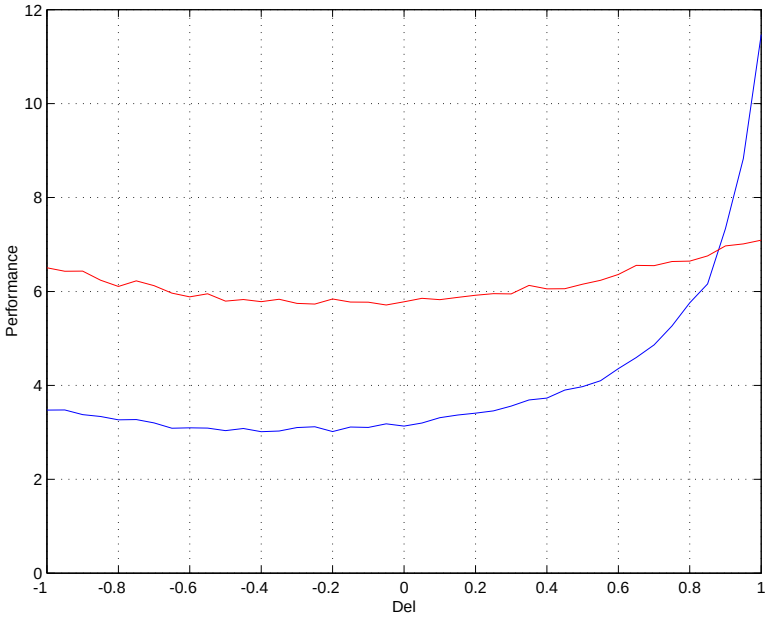

Figure 7.4: The performance of LEQG and LQG with damping. The top line is the LEQG cost; the lower line is the LQG cost

Results

The simulation was run for this form of perturbation, and as can be seen from Fig. 7.4, when $\Delta \to 1$, and energy is being added into the system, the LEQG filter begins to outperform the LQG filter. This can be seen by the dramatic rise in the performance criteria for the LQG (drop in performance), while the LEQG maintains a relatively stationary performance criterion.

To confirm that the LEQG filter is performing better, a plot of the atomic motion for the LQG and LEQG filters is provided in Fig.7.5. One can see that the LEQG filter has tighter constrained the atom to the origin, and had displayed a typical 'robust' property by denying large deviations. This is a classic LEQG idea stemming from the exponentiation of the cost function.

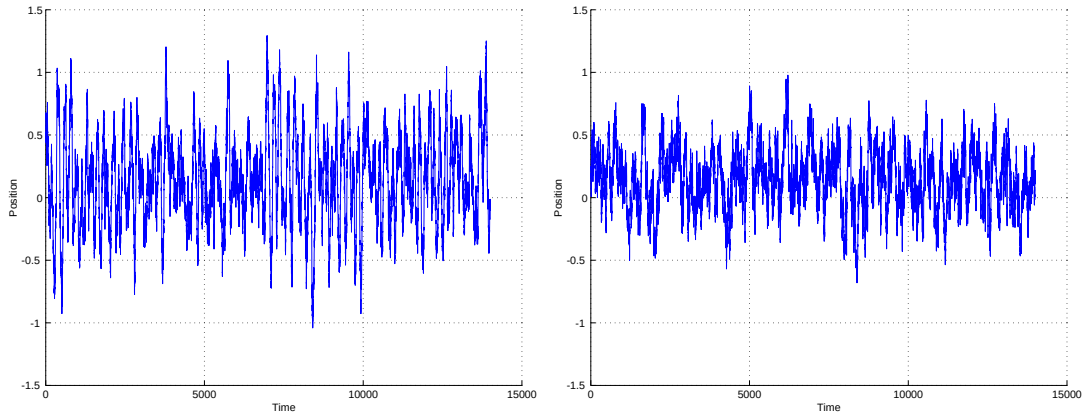

Figure 7.5: Sample paths for the atomic position. The LQG path is on the left, the LEQG path is on the right

Analysis

Although the graphs above have demonstrated a model where the LEQG filter performs better then the LQG filter, it is now necessary to establish a physical situation where such might occur. Since antifriction implies energy being added into the system, a mechanism for adding energy needs to determined.

To add energy into a quantum system is not so trivial. To begin with, one might consider adding terms to the Hamiltonian. However, it is necessary that the Hamiltonian implies a constant energy; that is its definition. The only term that can be added which is known to maintain the gaussian shape of the state [11] is one of the form $H_{af} = \alpha(xp + px)$. It must be a symmetric form to maintain the Hermitian property of the Hamiltonian. This will add a term to the dynamics of $d\langle x\rangle$ of the form $\langle [x, H_{af}]\rangle$ and similarly to $d\langle p\rangle$ at term of the form $\langle [p, H_{af}]\rangle$. It seem promising going through the calculation that the equations (See Eq.3.17, Eq.3.18) of the dynamics are modified by

$$d\langle x\rangle \to d\langle x\rangle + i\hbar 2\alpha\langle x\rangle\, dt$$

$$d\langle p\rangle \to d\langle p\rangle - i\hbar 2\alpha\langle p\rangle\, dt$$

however it can be seen that this does not introduce any velocity dependent term, but a modification of the form

$$\ddot{x} \to \ddot{x} + \alpha x$$

supporting the idea that any modification of the Hamiltonian will not introduce a term of the desired form.

Thus the other option is to modify the master equation. As elucidated earlier in

this thesis, the most common way of removing energy from a system is by employing the "damping" superoperator $\mathcal{D}[A]$ for some operator A given by

$$2\mathcal{D}[A]\rho = 2A\rho A^\dagger - A^\dagger A\rho - \rho A^\dagger A \tag{7.2}$$

This is used to describe spontaneous emission of an atom, or the decay of light out of a cavity. One possibility is to change the sign of the operator to produce an 'anti' damping term, introducing energy into the system.

Such a situation could be achieved, with a term for in $d\langle x\rangle$ proportional to

$$\langle \mathrm{Tr}\left(A^\dagger[A,x] + [x,A^\dagger]A\right)\rangle$$

and similarly for p. Obviously, the terms x and p won't work; neither will any Hermitian operators, such as $xp + px$, due to the cyclic property of the trace.

Providing such energy could come from an excited reservoir or heat bath, however unlikely. Further consideration of this type of equation, and modification of the master equation, may provide some light on the situation.

7.4 Conclusion

For the two situations illustrated above, two different results were found.

For Zeeman shift, a linear potential is added to the physical model as a perturbation again the nominal model in the controller. Under this situation the LQG filter performs better regardless of the strength of the perturbation. Thus the LEQG does not provide a sense of robustness against stray magnetic fields in an experimental situation that causes such a Zeeman shift.

For the situation of a spurious 'anti-friction' term, where energy is being coupled into the system, the LEQG performs better for large values of the perturbation. Unfortunately, so far no physically realistic mechanism for such a system is evident.

Chapter 8

Conclusion

In the situations of a quantum system under observation, various levels of approximations are required to produce the dynamical equations for the system. The resulting model, coupled with an appropriate cost function, provide the necessary requisites for an LQG analysis, with the resulting Kalman filter offering considerable control over the system. This involves dealing with back-action and classical measurement noise, to provide a best estimate of the system, and when coupled with the optimal controller, provides a relatively powerful feedback system.

The goal of this thesis is to explore the possible application of a new form of filter/controller, the risk-sensitive LEQG filter, which has been shown to enjoy a sense of robustness against model uncertainties. An investigation has been launched into some simple models, and a computer simulation has highlighted circumstances where the LEQG filter does provide better control. To this extent, this thesis has been successful.

There lies a certain irony into simulating robust quantum control theory on a computer. One must conceive of a physical mechanism that produces a perturbation to a system in an experimental situation, that would be over-looked or ignored for the control and feedback of the system. It is only under these circumstances that LEQG simulations will show better performance. In a real experimental situation, implementing LEQG over LQG provides only greater robustness by providing risk-sensitive control, over risk-neutral control.

Furthermore, this project has highlighted the major goals of further work into this area. Introducing non-linearities requires a higher level of computational power, that is not easily implemented using standard mathematical tools. XMDS provides a excellent machine for simulating many complex, non-linear, stochastic quantum system; it currently is not designed to deal with state-estimation or feedback loops, and the nature of measurement theory, expressed as a stochastic master equation, differs mathematically from the random nature of coupling to the environment. By coupling to the environment, a damping term can be added to a master equation, but no randomness is introduced to the master equation. Further work is inevitable.

Appendix 1

Below are the programs used to implement the simulation. Some have been left out due to there display in the main text of the thesis. This code is an example for the Anti friction case, but modifications for the Zeeman effect have been noted in the main text.

initLEQG.m

```
% initLEQG.m
% The P & Q for performance criteria
P = [mass*w^2,0;0,1/mass];
q = 1;
Q = q^2*P;

% The LEQG filter which models the physical system
A_c = [0,1/mass ;-mass*w^2,0];
B_c = eye(2);
N_c = 2*eta*k;
H_c = 0.5*8*eta*k*[1,0];
G_c = sqrt(2*k)*hbar*[0;1];

if mu == 0
    Y_c = care(A_c',H_c',G_c*G_c', N_c);
disp('mu = 0!!')
else
    B1_c = sqrt(P);
    B2_c = sqrt(H_c'*(N_c^-1)*H_c);
    B1 = [ B1_c , B2_c ]  ;
    m1 = size(B1_c,2);
    m2 = size(B2_c,2);
    R1 = [ -mu^-1*eye(m1), zeros(m1,m2);zeros(m2,m1), eye(m2) ];
    Y_c = care(A_c', B1,G_c*G_c', R1);
```

```
end

% The controller
if mu == 0
    X_c = care(A_c,B_c,P, Q);
else
    B1_c = sqrt(G_c*G_c');
    B2_c = sqrt(B_c*inv(Q)*B_c');
    B2 = [ B1_c , B2_c ] ;
    m1 = size(B1_c,2);
    m2 = size(B2_c,2);
    R2 = [ -mu^-1*eye(m1), zeros(m1,m2);zeros(m2,m1), eye(m2) ];
    X_c = care(A_c, B2, P, R2);
end
K = -inv(Q)*B_c'*X_c*inv(eye(2) - mu*Y_c*X_c);
```

initLQG.m

```
% initLQG.m

% The P & Q for performance criteria
P = [mass*w^2,0;0,1/mass];
q = 1;
Q = q^2*P;

% The LQG filter which models the physical system
A_c = [0,1/mass ;-mass*w^2,0];
B_c = eye(2);
N_c = 2*eta*k;
H_c = 0.5*8*eta*k*[1,0];
G_c = sqrt(2*k)*hbar*[0;1];
Y_c = care(A_c',H_c',G_c*G_c', N_c);

% The controller
X_c = care(A_c,B_c,P,Q);

K = -inv(Q)*B_c'*X_c;
```

phys_init.m

```
% Initialize the information for physical constants and the
mass = 1;
```

```
w = 2;
eta = 0.9;
k = w;  % w = ck, and c = 1
hbar = 1;

% The physical system - note H_p and N_p are
%  measurement parameters, but cannot be separated
A_p = [0 ,1/mass ;-mass*w^2,0];
B_p = eye(2);
G_p = sqrt(2*k)*hbar*[0;1];
% For measurement
H_p = 0.5*8*eta*k*[1,0];
N_p = 2*eta*k;
% Variances
Y_p = care(A_p',H_p',G_p*G_p');
```

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