Effective super Tonks-Girardeau gases as ground states of strongly attractive multicomponent fermions

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In the strong interaction limit, attractive fermions with N-component hyperfine states in a one-dimensional waveguide form unbreakable bound cluster states. We demonstrate that the ground state of strongly attractive SU(N) Fermi gases can be effectively described by a super Tonks-Girardeau gaslike state composed of bosonic cluster states with strongly attractive cluster-cluster interaction for even N and a Fermi duality of a super Tonks-Girardeau gaslike state composed of fermionic cluster states with weakly interacting cluster-cluster p-wave interaction for odd N.

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I. INTRODUCTION

The experimental progress on trapping ultracold atoms in tightly confined waveguides in a well-controlled way [1–4] has stimulated intensive study of the physical properties of one-dimensional (1D) quantum gases. The effective interaction strength between atoms in a 1D waveguide can be tuned via Feshbach resonance or confinement-induced resonance [5], leading to the experimental realization of Tonks-Girardeau (TG) gases [3,4]. The TG gas describes the strongly repulsive Bose gas [6,7]. Starting from the TG gas and then switching the interaction between atoms from strongly repulsive to strongly attractive, the experimental realization of a 1D *super* Tonks-Girardeau (STG) gas of bosonic cesium atoms was reported very recently [8]. In contrast to the TG gas, the STG gas describes a gaslike phase of the attractive Bose gas [9–11], which is metastable against falling into its cluster-type ground state [12], despite the fact that the interaction between atoms is strongly attractive [13,14].

Although the STG gas realized in Ref. [8] is a metastable highly excited state of the attractive Bose gas, in recent theoretical work [15] it was found that the ground state of a strongly attractive spin-1/2 Fermi gas can be effectively described by the STG gas. Intuitively, two fermions with opposite spins form a tightly bound state and the bound pairs of fermions can be viewed as composite bosons with a mass of $2m_F$. The effective interaction between the composite bosons is also attractive with the interaction strength given by $c_B = 2c_F$ [15,16]. Conversely, the ground state of the bound Fermi pairs is described by the STG phase of attractive bosons [15]. Very recently, multicomponent Fermi gases have attracted considerable interest [17–19] due to the novel existence of different sizes of molecules. Here we consider the interesting question of whether the ground state of a strongly attractive multicomponent fermionic system can be also effectively described by an STG gas of multiparticle bound states. A positive answer is confirmed by explicit identification of the general mapping relation between the attractive SU(N) Fermi gas and the STG gas.

II. ATTRACTIVE N-BOUND FERMIONS

We consider a δ-function interacting system composed of N_F atomic attractive fermions with equal mass m_F which occupy N hyperfine levels with identical particle numbers $N^i = N_N = N_F/N$ $(i = 1,\dots,N)$ and constrained by periodic boundary conditions to a line of length L. If the interactions are spin independent, the Hamiltonian reads

$$H_F = \sum_{i=1}^{N_F} -\frac{\hbar^2}{2m_F}\frac{\partial^2}{\partial x_i^2} + g_F \sum_{i<j} \delta(x_i - x_j), \qquad (1)$$

where $g_F = -2\hbar^2/(m_F a_{1D}^F)$ is the interaction strength with a_{1D}^F the 1D effective s-wave scattering length. Although the interactions in Hamiltonian (1) are represented in terms of δ interaction, the exchange antisymmetric wave function for fermions gives the restriction that interactions among the same fermion level are prohibited. Different symmetries of the wave function produce different Bethe ansatz equations even if the Hamiltonian has the same form as Eq. (1) [20,21]. For simplicity, we use the dimensionless coupling constant $\gamma_F = c_F/n_F$ with density $n_F = N_F/L$ and $c_F = -2/a_{1D}^F$. In the strongly attractive limit ($g_F \to -\infty$), atoms form tightly bound states with each bound state composed of N fermions in different hyperfine states [22]. No tightly bound state with more than N fermions can be formed in an N-component Fermi gas due to the Pauli exclusion principle.

If N is even, the tightly bound state can be viewed as a composite boson with a mass $m_B = Nm_F$. In this work, we find that the ground state of the strongly attractive SU(N) Fermi gas with N even can be effectively described by a STG gas of attractive composite bosons. This can be viewed as a direct generalization of the SU(2) Fermi gas result [15]. On the other hand, if N is odd, the tightly bound state can only be viewed as a composite fermion. Obviously, strongly attractive SU(N) fermions with N odd cannot be mapped to an effective bosonic gas as for the two-component case [15]. Nevertheless, we shall show that it can be mapped to a spinless Fermi gas with weakly interacting p-wave interaction. Due to a general

Fermi-Bose mapping [23], eigenstates of a spinless Fermi gas with p-wave interaction of any strength can be mapped to those of a 1D Bose gas with δ-function interactions. Therefore the gaslike state of a weakly repulsive p-wave Fermi gas can be viewed as the Fermi correspondence of the STG phase of a strongly attractive Bose gas.

Before construction of the mapping between the ground state of the strongly attractive SU(N) Fermi gas and the STG phase of composite bosons or fermions, we first discuss the solution of the SU(N) Fermi gas which is exactly solvable by the Bethe ansatz method. The eigenvalues of Hamiltonian (1) are given by

$$E = \frac{\hbar^2}{2m_F}\sum_{j=1}^{N_F} k_j^2 \tag{2}$$

with quasimomentum k_j determined by the Bethe ansatz equations (BAEs) [20–22]

$$\exp(ik_jL) = \prod_{\alpha=1}^{M_1}\frac{k_j - \Lambda_\alpha^{(1)} + ic'_F}{k_j - \Lambda_\alpha^{(1)} - ic'_F}, \tag{3}$$

$$\prod_{\beta=1}^{M_{\ell-1}}\frac{\Lambda_\alpha^{(\ell)} - \Lambda_\beta^{(\ell-1)} + ic'_F}{\Lambda_\alpha^{(\ell)} - \Lambda_\beta^{(\ell-1)} - ic'_F} = -\prod_{\gamma=1}^{M_\ell}\frac{\Lambda_\alpha^{(\ell)} - \Lambda_\gamma^{(\ell)} + ic_F}{\Lambda_\alpha^{(\ell)} - \Lambda_\gamma^{(\ell)} - ic_F} \times\prod_{\nu=1}^{M_{\ell+1}}\frac{\Lambda_\alpha^{(\ell)} - \Lambda_\nu^{(\ell+1)} - ic'_F}{\Lambda_\alpha^{(\ell)} - \Lambda_\nu^{(\ell+1)} + ic'_F}, \tag{4}$$

for $j = 1,\dots,N_F$, $\alpha = 1,\dots,M_\ell$ and $\ell = 1,\dots,N-1$. We have denoted $M_0 = N_F$ and $\Lambda_j^{(0)} = k_j$. The parameters $\{\Lambda_\alpha^{(\ell)}\}$ are the spin rapidities. The quantum numbers are given by $M_l = (N-l)N_N$ and $c'_F = c_F/2$.

For strongly attractive attraction, i.e., $L|c_F| \gg 1$, the BAEs permit different sizes of charge bound state. As a consequence of the Pauli exclusion principle and the SU(N) symmetry, there is no tightly bound state with more than N fermions for the SU(N) Fermi gas [24]. For the ground state, there are equal numbers of particles in each hyperfine spin state. In this state, the charge bound state in k space is of the form

$$k_{q,j} = \Lambda_q^{(N-1)} + (N+1-2j)\,c'_F + O(i\delta|c_F|), \tag{5}$$

for $j = 1,2,\dots,N$ and $q = 1,2,\dots,N_F/N$. The spin rapidities form a certain pattern of spin string solutions. For the ground state, each charge bound state $k_{q,j}$ with different q is accompanied by a spin string $\{\Lambda_{q,\alpha}^{(1)}\}$ of length $N-1$ with $\alpha = 1,2,\dots,N-1$, a spin string $\{\Lambda_{q,\alpha}^{(2)}\}$ of length $N-2$ with $\alpha = 1,2,\dots,N-2$, and so on, until the real root $\Lambda_q^{(N-1)}$ in the last spin branch [22]. In this special case, the spin strings read

$$\Lambda_{q,\alpha}^{(r)} = \Lambda_q^{(N-1)} + i(N-r+1-2\alpha)\,c'_F + O(i\delta|c_F|), \tag{6}$$

with $\alpha = 1,\dots,N-r$ for $r = 1,\dots,N-2$, respectively. In the above equations δ is a very small number of order $\exp(-L|c_F|)$.

Substituting the charge bound states $k_{q,j}$ with $j = 1,2,\dots,N$ and the spin strings into Eq. (3) results in N equations. After multiplying these N equations together and combining with Eq. (4) (see Appendix) the BAEs reduce to

$$\exp(Ni\Lambda_qL) = (-1)^{N_F-1}\prod_{\beta=1}^{N_N}\prod_{r=1}^{N-1}\frac{\Lambda_q - \Lambda_\beta + irc_F}{\Lambda_q - \Lambda_\beta - irc_F}. \tag{7}$$

The eigenvalues of Hamiltonian (1) are then given by

$$E = -N_N\epsilon_b + \frac{\hbar^2}{2m_F}\sum_{q=1}^{N_N} N\Lambda_q^2 \tag{8}$$

with binding energy $\epsilon_b = (\hbar^2/2m_F)N(N^2-1)\,c_F^2/12$.

In the strongly attractive limit and in the absence of an external field, the N-fermion clusters are unbreakable and we may subtract the binding energy from the energy, i.e.,

$$E_F = E + N_N\epsilon_b = \frac{\hbar^2}{2m_F}\sum_{q=1}^{N_N} N\Lambda_q^2, \tag{9}$$

which includes the kinetic energy of the bound clusters and the interaction energy produced from cluster-cluster scattering.

In the thermodynamic limit, $N_F \to \infty$ and $L \to \infty$ at fixed density n_F, the energy of the system can be represented in the integral form

$$E_F = \frac{\hbar^2 L}{2m_F}N\int_{-B}^{B}\Lambda^2\rho_F(\Lambda)\,d\Lambda, \tag{10}$$

where $\rho_F(\Lambda)$ is the density distribution for Λ determined by the integral form of BAEs (7) as

$$\rho_F(\Lambda) = \frac{N}{2\pi} - \frac{1}{\pi}\sum_{r=1}^{N-1}\int_{-B}^{B}\frac{r|c_F|}{r^2c_F^2 + (\Lambda-\Lambda')^2}\rho_F(\Lambda')\,d\Lambda'. \tag{11}$$

The integration limit B is determined by the linear density $n_F = N\int_{-B}^{B}\rho_F(\Lambda)d\Lambda$.

In terms of the dimensionless energy $e_N(\gamma_F)$, we have

$$E_F = \frac{\hbar^2 L}{2m_N}n_N^3 e_N(\gamma_F), \tag{12}$$

where $n_N = n_F/N$, $m_N = Nm_F$ and

$$e_N(\gamma_F) = \frac{N^6|\gamma_F|^3}{\lambda^3}\int_{-1}^{1} z^2 g_N(z)\,dz. \tag{13}$$

Here we have defined $z = \Lambda/B$, $\lambda = |c_F|/B$ and $g_N(z) = \rho(Bz)/N$. The scaled density distribution $g_N(z)$ is determined by

$$g_N(z) = \frac{1}{2\pi} - \frac{1}{\pi}\sum_{r=1}^{N-1}\int_{-1}^{1}\frac{r\lambda}{r^2\lambda^2 + (z-z')^2}g_N(z')\,dz',$$

$$\lambda = N^2|\gamma_F|\int_{-1}^{1} g_N(z)\,dz,$$

which come from Eq. (11) and the linear density n_F. In the strongly attractive limit with $|\gamma_F| \gg 1$, we can expand the dimensionless energy in terms of $1/|\gamma_F|$. Up to third order, this gives

$$E_F = \frac{\hbar^2 N_N^3}{2m_N L^2}\frac{\pi^2}{3}\left[1+\frac{4}{|\gamma_N|}+\frac{12}{\gamma_N^2}+\frac{32}{|\gamma_N|^3}\left(1-\frac{\pi^2\eta}{15\zeta^3}\right)\right], \tag{14}$$

in which $\gamma_N = N^2\gamma_F/\zeta$, with $\zeta = \sum_{r=1}^{N-1} 1/r$ and $\eta = \sum_{r=1}^{N-1} 1/r^3$.

A. Equivalence to a super TG gas for N even

The unbreakable N-fermion cluster state is effectively described as a composite boson for even N. Before constructing the mapping relation, we first give a brief review of the STG state of the attractive Bose gas. The 1D interacting Bose gas composed of N_B bosons with mass m_B is described by the Hamiltonian

$$H_B = \sum_{i=1}^{N_B} -\frac{\hbar^2}{2m_B}\frac{\partial^2}{\partial x_i^2} + g_B\sum_{i<j}\delta(x_i - x_j), \tag{15}$$

with interaction $g_B = -2\hbar^2/(m_B a_{1D}^B)$, where a_{1D}^B is the 1D s-wave scattering length. The energy eigenvalues are given by

$$E_B = \frac{\hbar^2}{2m_B}\sum_{j=1}^{N_B} k_j^2, \tag{16}$$

where the k_j are determined by the BAE [7]

$$\exp(ik_jL) = -\prod_{l=1}^{N_B}\frac{k_j - k_l + ic_B}{k_j - k_l - ic_B}, \tag{17}$$

with $c_B = m_B g_B/\hbar^2 = -2/a_{1D}^B$.

For attractive bosons, the ground-state solution for the BAE (17) is a complex string solution corresponding to McGuire's cluster state [12]. On the other hand, the BAE (17) has real solutions even for $c_B < 0$, which correspond to some highly excited states of the attractive Bose gas. The super TG state is the lowest gaslike state with real solutions for BAE (17) [11,13,15]. In the strongly attractive limit, the energy of the STG state of the attractive Bose gas can be expressed as

$$E_{\rm STG} = \frac{\hbar^2 N_B^3}{2m_B L^2}\frac{\pi^2}{3}\left[1+\frac{4}{|\gamma_B|}+\frac{12}{|\gamma_B|^2}+\frac{32}{|\gamma_B|^3}\left(1-\frac{\pi^2}{15}\right)\right] \tag{18}$$

with $\gamma_B = c_B/n_B$.

Comparing Eqs. (14) and (18), it is clear that the two expressions are identical up to the second order of γ_F if $\gamma_B = \gamma_N = N^2\gamma_F/\zeta$, $N_B = N_N = N_F/N$, and $m_B = m_N = Nm_F$. Since the N-bound state formed by N fermions with opposite spin for even N has a mass $m_B = Nm_F$, we can conclude that the N_N N-bound states are equivalently described by the super-TG phase of the interacting Bose gas with the effective 1D scattering length

$$a_{1D}^B = \frac{\zeta}{N}a_{1D}^F. \tag{19}$$

Schematically, we illustrate such a mapping in Fig. 1(a) by taking the SU(4) Fermi gas as an example. We also compare the ground-state energy of the SU(N) ($N = 2$,4) Fermi gas with the energy of the STG phase of the Bose gas composed of composite bosons with mass Nm_F in Fig. 2. The ground-state energy of the SU(2) Fermi gas is identical to the energy of the STG phase of the Bose gas for all γ_N [15], whereas the ground-state energy of the SU(4) Fermi gas matches very well to that of the STG gas for large γ_N. As shown in Fig. 3, the relative error for the ground-state energy of the SU(N) Fermi gas and the energy of the corresponding STG gas is less than 0.1 for $|\gamma_N| = 10$ and less than 10^{-7} for $|\gamma_N| = 600$. This indicates that the STG gas provides a good effective description for the ground state of the strongly attractive SU(N) Fermi gas, although the mapping is not exact for all γ_N like for the SU(2) case.

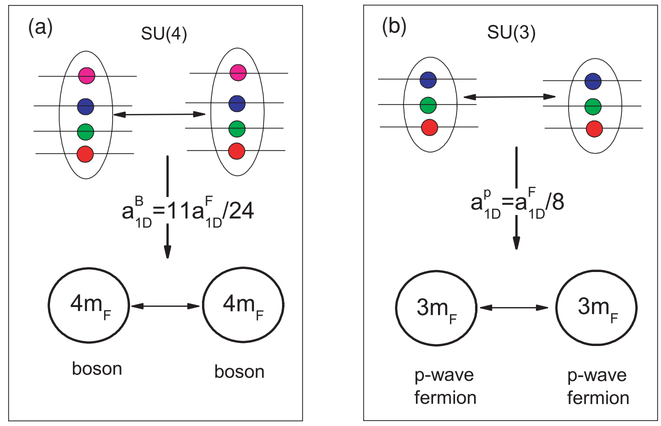

FIG. 1. (Color online) The strongly attractive N-bound state Fermi gas can be effectively described by a super Tonks-Girardeau gas composed of attractive bosons for (a) even N and can be effectively described by a super Fermi Tonks-Girardeau gas composed of p-wave repulsive fermions for (b) odd N.

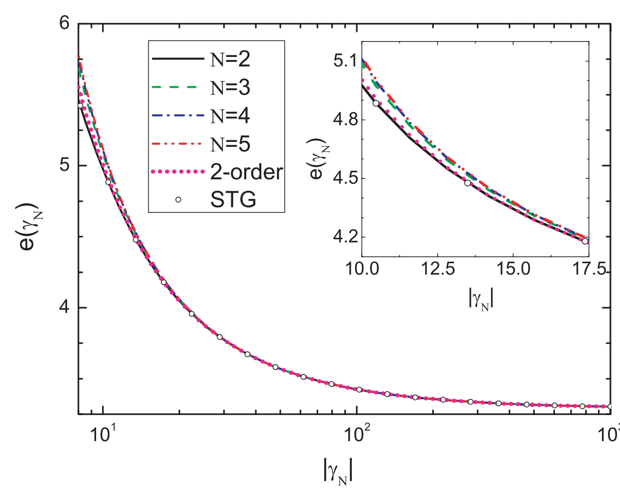

FIG. 2. (Color online) Comparison of the ground-state energies of the attractive SU(N) Fermi gas and the energy of the corresponding STG phase of bosons (for even N) or p-wave fermions (for odd N) (effective repulsive p-wave fermions) for different N with large interaction γ_N. The inset shows a magnified view of ground-state energies.

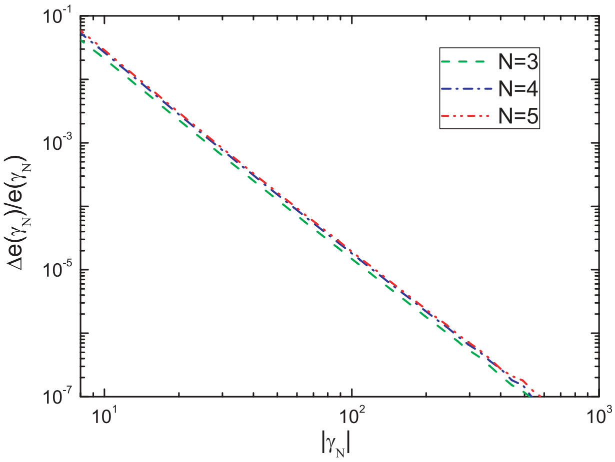

FIG. 3. (Color online) The relative error vs $|\gamma_N|$, where $\Delta e(\gamma_N) = e(\gamma_N) - e_{\rm STG}(\gamma_N)$.

B. Equivalence to a super Fermi TG gas for N odd

The Hamiltonian for the 1D p-wave interacting polarized Fermi gas reads

$$H_p = \sum_{i=1}^{N_p} -\frac{\hbar^2}{2m_p}\frac{\partial^2}{\partial x_i^2} + g_p \sum_{i<j} V(x_i - x_j), \qquad (20)$$

where $V(x_i - x_j) = (\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j})\delta(x_i - x_j)(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j})$ is the pseudopotential for p-wave interaction [25–27] and $g_p = -2\hbar^2 a_{\rm 1D}^p/m_p$ [25,26]. The dimensionless interaction parameter is defined by $\gamma_p = m_p g_p n_p/\hbar^2 = -2a_{\rm 1D}^p n_p$. For p-wave interacting fermions, the energy eigenvalues are given by

$$E_p = \frac{\hbar^2}{2m_p}\sum_{j=1}^{N_p} k_j^2, \qquad (21)$$

where the quasimomenta k_j are determined by the BAE [28,29]

$$\exp(ik_jL) = -\prod_{l=1}^{N_p}\frac{k_j - k_l + ic_p}{k_j - k_l - ic_p}, \qquad (22)$$

where the parameter $c_p = -1/(2a_{\rm 1D}^p)$.

It is clear that BAE (22) is identical to BAE (17) if $c_p = c_B$ and $N_p = N_B$ [28,29], which means that there is a one-to-one correspondence between the p-wave Fermi gas and the interacting Bose gas [23]. Correspondingly, the STG state of the attractive Bose gas has a Fermi correspondence which is the lowest gaslike state of the weakly interacting p-wave fermions with $g_p \to 0^+$. For weakly interacting p-wave fermions in the thermodynamic limit, the energy of the lowest gaslike state has the form

$$E_p = \frac{\hbar^2}{2m_p}\frac{N_p^3}{L^2}\frac{\pi^2}{3}\left[1 + 4|\gamma_p| + 12|\gamma_p|^2 + 32\left(1 - \frac{\pi^2}{15}\right)|\gamma_p|^3 + \cdots\right], \qquad (23)$$

where $|\gamma_p| \ll 1$.

Comparing Eqs. (14) and (23), it is clear that the two expressions are identical up to the second order in γ_F if $\gamma_p = 1/\gamma_N = \zeta/(N^2\gamma_F)$, $N_p = N_N = N_F/N$, and $m_p = m_N = Nm_F$. Since the N-bound state formed by N fermions with different hyperfine states for odd N is a composite fermion with a mass $m_p = Nm_F$, we can conclude that the N_N N-bound states are equivalently described by the gaslike state of the weakly interacting p-wave Fermi gas with the effective 1D scattering length

$$a_{\rm 1D}^p = \frac{\zeta a_{\rm 1D}^F}{4N}. \qquad (24)$$

The mapping is schematically displayed in Fig. 1(b). The comparison of the energies of the SU(N) ($N = 3,5$) Fermi gas and the STG phase of the p-wave Fermi gas is also given in Fig. 2, which indicates a good matching in the limit of large $|\gamma_N|$. Similarly, as shown in Fig. 3, the relative error for the ground-state energy of the SU(N) Fermi gas and the energy of the corresponding STG state of the p-wave Fermi gas is less than 10^{-7} for $|\gamma_N| = 600$. This indicates that, although not exact for all γ_N, the mapping provides a very good description for large $|\gamma_N|$.

III. SUMMARY

In summary, we have examined the equivalence between the ground state of the strongly attractive N-component Fermi gas and the super Tonks-Girardeau phase of an effective Bose or p-wave Fermi gas. By comparing the ground-state energy of strongly attractive fermions with the energy of the super Tonks-Girardeau phase of the Bose gas or p-wave Fermi gas, we find that the bound N-fermion clusters formed in the strongly attractive regime should be described by the super Tonks-Girardeau phase of attractive composite bosons (for even N) or composite fermions with effective p-wave interactions (for odd N). The super Tonks-Girardeau gas phase thus provides an effective description for the ground state of strongly attractive multicomponent fermions.

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APPENDIX: DERIVATION OF BETHE ANSATZ EQUATIONS FOR FERMI BOUND STATES

Here we show how to derive BAEs for Fermi bound states by taking as example the system of three-component fermions. For simplicity, we define the function

$$e_n(x) = \frac{x + inc'_F}{x - inc'_F}.$$

For three-component fermions the BAEs (3) and (4) are

$$e^{ik_jL}=\prod_{\alpha=1}^{M_1}e_1\big(k_j-\Lambda_\alpha^{(1)}\big), \tag{A1}$$

$$\prod_{j=1}^{N_F}e_1\big(\Lambda_\alpha^{(1)}-k_j\big)=-\prod_{\beta=1}^{M_1}e_2\big(\Lambda_\alpha^{(1)}-\Lambda_\beta^{(1)}\big)\times\prod_{l=1}^{M_2}e_{-1}\big(\Lambda_\alpha^{(1)}-\Lambda_l^{(2)}\big), \tag{A2}$$

$$\prod_{\alpha=1}^{M_1}e_1\big(\Lambda_l^{(2)}-\Lambda_\alpha^{(1)}\big)=-\prod_{m=1}^{M_2}e_2\big(\Lambda_l^{(2)}-\Lambda_m^{(2)}\big) \tag{A3}$$

for $j=1,2,\ldots,N_F$, $\alpha,\beta=1,2,\ldots,M_1$, and $l,m=1,2,\ldots,M_2$. Here we confine our attention to the equally populated case $N_F=3N_3$, $M_1=2N_3$, and $M_2=N_3$.

For strong attraction, i.e., for $L|c_F|\gg 1$, the charge bound states and spin strings are of the form

$$k_{q,h_1}=\Lambda_q+i(4-2h_1)c_F'+O(i\delta|c_F|), \tag{A4}$$

$$\Lambda_{q,h_2}^{(1)}=\Lambda_q+i(3-2h_2)c_F'+O(i\delta'|c_F|), \tag{A5}$$

for exponentially small δ and δ', with $\Lambda_q=\Lambda_q^{(2)}$, $h_1=1,2,3$, $h_2=1,2$, and $q=1,2,\ldots,N_3$.

For the attractive regime, the common real parts in the bound states (A4) and spin strings (A5) lead to zero factors in the BAEs (A1)–(A3). In order to avoid ill-defined equations, we eliminate such zero factors in the BAE level by level. The first step is to deal with the charge bound state k_{q,h_1} with $h_1=1,2,3$ in the BAE (A1), i.e.,

$$e^{i3\Lambda_qL}=e^{i(k_{q,1}+k_{q,2}+k_{q,3})L}=\prod_{h_2=1}^{2}\prod_{h_1=1}^{3}e_1\big(k_{q,h_1}-\Lambda_{q,h_2}^{(1)}\big)\times\prod_{\alpha=1}^{M_1/2}e_2(\Lambda_q-\Lambda_\alpha)e_4(\Lambda_q-\Lambda_\alpha) \tag{A6}$$

The first terms on the right-hand side of (A6) contain zero factors which have to be eliminated. From the second BAE (A2), we have

$$\prod_{h_2=1}^{2}\prod_{h_1=1}^{3}e_{-1}\big(k_{q,h_1}-\Lambda_{q,h_2}^{(1)}\big)\prod_{\alpha=1}^{N_F/3}e_2(\Lambda_q-\Lambda_\alpha)e_4(\Lambda_q-\Lambda_\alpha)=\prod_{h_2=1}^{2}e_{-1}\big(\Lambda_{q,h_2}^{(1)}-\Lambda_q\big)\prod_{\alpha=1}^{M_1/2}e_2(\Lambda_q-\Lambda_\alpha)e_4(\Lambda_q-\Lambda_\alpha). \tag{A7}$$

In order to eliminate the first factor on the right-hand side of these equations, we extract this factor from the third BAE (A3), i.e.,

$$\prod_{h_2=1}^{2}e_{-1}\big(\Lambda_{q,h_2}^{(1)}-\Lambda_q\big)\prod_{\alpha=1}^{M_1/2}e_2(\Lambda_q-\Lambda_\alpha)=\prod_{m=1}^{M_2}e_2(\Lambda_l-\Lambda_m). \tag{A8}$$

Substituting (A8) into (A7), Eq. (A6) becomes

$$e^{i3\Lambda_qL}=\prod_{\alpha=1}^{M_1/2}e_2(\Lambda_q-\Lambda_\alpha)e_4(\Lambda_q-\Lambda_\alpha), \tag{A9}$$

which is the three-component fermion form of Eq. (7).

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