MHD WAVES AND GLOBAL ALFVÉN EIGENMODES

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OUTLINE OF LECTURE 4

• Linearised MHD equations
• MHD waves
• Compressional waves
• Shear Alfvén waves
• Kinetic Alfvén waves
• Global Alfvén Eigenmodes
• Summary
ALFVÉN INSTABILITIES DRIVEN BY ALPHA-PARTICLES

• Alpha-particles (He$^4$ ions) are born in deuterium-tritium nuclear reactions with birth energy 3.52 MeV. These fusion-born ions are super-Alfvénic,

$$V_{ Ti} \sim 10^6 \text{ m/s} \ll V_A \sim 5 \cdot 10^6 \text{ m/s} < V_\alpha = 1.3 \cdot 10^7 \text{ m/s} \ll V_{ Te} \sim 8 \cdot 10^7 \text{ m/s},$$

where $V_A = B_0/\left(\mu_0 n_i m_i\right)^{1/2}$, and the estimate is for D:T=50:50 ITER plasmas.

• During slowing-down of alpha-particles, they pass the resonance condition $V_A = V_{\parallel \alpha}$ and may excite Alfvén waves with $\omega = k_{\parallel} V_A$ dispersion relation.

• Free energy source: radial gradient of alpha-particle pressure. The instability results in radial re-distribution of alpha-particles.

• Re-distribution may also cause losses of highly energetic alphas → damage to the first wall.

• We have to assess possible wave-particle interaction in burning plasmas.
STARTING MHD EQUATIONS

- For describing *plasma particles*, we take velocity moments of the kinetic equations for electron and thermal ion distribution functions and obtain

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0; \]
\[
\rho \frac{d \mathbf{V}}{dt} = -\nabla p + \frac{1}{c} \mathbf{J} \times \mathbf{B};
\]
\[
\frac{\partial p}{\partial t} + \mathbf{V} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{V} = 0;
\]
\[
\mathbf{E} + \frac{1}{c} \mathbf{V} \times \mathbf{B} = 0;
\]

- For describing * electromagnetic fields* in the plasma, Maxwell’s equations are used

\[
\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J};
\]
\[
\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t};
\]
\[
\nabla \cdot \mathbf{B} = 0.
\]

- Here, scale lengths larger than Debye length are considered with \( n_e = \sum_i Z_i \cdot n_i \)
THE LINEARISATION PROCEDURE

- All the field and plasma variables are represented as sums of equilibrium (denoted by subscript 0) and perturbed (denoted by $\delta$) quantities:

$$J = J_0 + \delta J, \quad B = B_0 + \delta B, \quad V = \delta V, \quad p = p_0 + \delta p, \quad \rho = \rho_0 + \delta \rho, \quad E = \delta E,$$

(*)

where all the perturbed quantities satisfy $\delta << 1$, i.e. $|\delta J / J_0| << 1$ etc.

- Substitute the expressions (*) in the starting set of equations and obtain equations with terms:
  a) not having $\delta$ at all; b) having $\delta$; c) having $\delta^2$ etc.

- The terms NOT having $\delta$ are balanced thanks to the plasma equilibrium

$$\nabla p_0 = \frac{1}{c} J_0 \times B_0$$

The relation between equilibrium quantities $J_0$, $p_0$, $B_0$ MUST be kept in all equations with $\delta$

- All the equations are linearised then, i.e. only linear terms in $\delta$ are kept and terms with $\delta^2$ etc. are dropped off as small (since $\delta << 1$)
LINEARISED MHD EQUATIONS

- The linearised ideal MHD equations take the form:

\[
\frac{\partial \delta \rho}{\partial t} + \nabla \cdot (\rho_0 \delta \mathbf{V}) = 0; \\
\rho_0 \frac{d \delta \mathbf{V}}{dt} = -\nabla \delta p + \frac{1}{4\pi} [\nabla \times \delta \mathbf{B}] \times \mathbf{B}_0; \\
\frac{\partial}{\partial t} \delta \mathbf{B} = \nabla \times [\delta \mathbf{V} \times \mathbf{B}_0]; \\
\delta p = \gamma \frac{p_0}{\rho_0} \delta \rho; \\
\]

- Introduce plasma displacement from the equilibrium, \( \xi \), related to \( \delta \mathbf{V} \) via

\[
\delta \mathbf{V} = \partial \xi \partial t \\
\]

- From the first and third equations we find then

\[
\delta \rho = -\text{div}(\rho_0 \xi); \\
\delta \mathbf{B} = \nabla \times [\xi \times \mathbf{B}_o] = -\mathbf{B}_o \text{div} \xi_\perp + \mathbf{B}_o \frac{\partial \xi_\perp}{\partial z} \\
\]

where we used \( \nabla \times [\mathbf{a} \times \mathbf{b}] = (\mathbf{b} \nabla) \mathbf{a} - (\mathbf{a} \nabla) \mathbf{b} + \mathbf{a} \text{div} \mathbf{b} - \mathbf{b} \text{div} \mathbf{a} \), and \( \mathbf{B}_o \uparrow \uparrow \mathbf{e}_z \)

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EQUATION FOR IDEAL MHD WAVES

- Substitute the expressions for $\delta \rho$, $\delta B$ in the remaining two equations and obtain

$$\frac{\partial^2 \xi}{\partial t^2} = c_s^2 \text{div} \xi + V_A^2 \text{div} V + V_A^2 \frac{\partial^2 \xi}{\partial z^2},$$

Where $c_s^2 = \gamma \, \frac{p_0}{\rho_0}$ is the ion sound speed, $V_A^2 = \frac{B_0^2}{4\pi \rho_0}$ is the Alfvén velocity

- This equation describes linear MHD perturbations of homogeneous ideally conducting plasma. Single vector equation gives three scalar equations for three types of waves
PLASMA DISPLACEMENT IN MHD WAVES

- **Compressional Alfvén and slow magnetosonic waves:** the “returning” force is the magnetic and the kinetic pressure

- **Shear Alfvén wave:** the “returning” force is the tension of magnetic field lines
COMPRESSIONAL WAVES - 1

- Coming back to the main equation

\[ \frac{\partial^2 \xi}{\partial t^2} = c_s^2 \nabla \text{div} \xi + V_A^2 \nabla_\perp \text{div} \xi_\perp + V_A^2 \frac{\partial^2 \xi_\perp}{\partial z^2} \quad (*) \]

- Consider two “compressible” types of waves, in which \( \xi_z \neq 0 \) and \( \text{div} \xi_\perp \neq 0 \)

- The parallel displacement \( \xi_z \) is described by the parallel projection of equation (\( (*) \)):

\[ \frac{\partial^2 \xi_z}{\partial t^2} = c_s^2 \frac{\partial^2 \xi_z}{\partial z^2} + c_s^2 \frac{\partial}{\partial z} \text{div} \xi_\perp \]

- We obtain equation for \( \text{div} \xi_\perp \) by taking divergence of perpendicular projection of \( (*) \):

\[ \frac{\partial^2 \text{div} \xi_\perp}{\partial t^2} = c_s^2 \Delta_\perp \text{div} \xi_\perp + V_A^2 \left( \Delta_\perp + \frac{\partial^2}{\partial z^2} \right) \text{div} \xi_\perp + c_s^2 \Delta_\perp \frac{\partial \xi_z}{\partial z} \]

Here, \( \Delta_\perp = \text{div} \nabla_\perp \)

- We see, that two equations for \( \xi_z \) and \( \text{div} \xi_\perp \) are coupled
COMPRESSIONAL WAVES - 2

- Consider limit $\beta \approx c_s^2 / V_A^2 \ll 1$. In this case, equation for $\text{div} \xi_\perp$ reduces to

$$\frac{\partial^2 \text{div} \xi_\perp}{\partial t^2} = V_A^2 \Delta \text{div} \xi_\perp$$

which decouples from $\xi_z$ and describes *Compressional Alfvén Wave*. The magnetic pressure $B_0^2 / 8\pi$ determines the “returning” force that acts *perpendicular* to $B_0$

- The displacement $\xi_z$ *parallel* to $B_0$ is described by

$$\frac{\partial^2 \xi_z}{\partial t^2} = c_s^2 \frac{\partial^2 \xi_z}{\partial z^2}$$

and such wave corresponds to *Ion Sound Wave* existing in plasma even without $B_0$

- However, if $\beta$ is *not* small, the decoupling of compressible waves does not work! The ion sound wave is modified then by the magnetic pressure and becomes *Slow Magnetosonic Wave*. The coupled equations give for $\xi_z$, $\text{div} \xi_\perp \propto \exp(-i\omega t + ik \cdot r)$ the following relation between wave frequency and wave vector (dispersion relation):

$$\omega^4 - (c_s^2 + V_A^2)k^2 \omega^2 + V_A^2 c_s^2 k_z^2 k^2 = 0$$
SHEAR ALFVÉN WAVES

- In contrast to the compressional waves, the third type of MHD waves, so-called Shear Alfvén wave, is incompressible:
  \[ \xi_z = 0 \quad \text{and} \quad \text{div} \, \xi_\perp = 0 \]

- For such waves the main MHD equation becomes simply
  \[ \frac{\partial^2 \xi_\perp}{\partial t^2} = V_A^2 \frac{\partial^2 \xi_\perp}{\partial z^2} \]

  which coincides with equation for string oscillations. The “returning” force is the tension of magnetic field lines, which act similarly to the strings

- In shear Alfvén wave the fluid displacement vector \( \xi \) and \( \vec{E} \) are perpendicular to the magnetic field \( B_0 \). The wave propagates along \( B_0 \):
  \[ \omega = \pm k_\parallel V_A \quad ; \quad V_A = \frac{B_0}{\sqrt{4\pi \sum_i n_i M_i}} \quad ; \quad k_\parallel = k \cdot B_0 / B_0 \]

- Among all the waves in plasmas, the Alfvén wave (H. Alfvén, Arkiv. Mat. Astron. Fysik 29B(2) (1942)) constitutes the most significant part of the MHD spectrum and is probably the best studied
HOW THIS WAVE EVOLVES IN INHOMOGENEOUS PLASMA?

The life-time of a wave-packet of shear Alfvén waves is limited by the “phase mixing”

\[ \tau^{-1} \propto \frac{d}{dr} \left( k_\parallel(r) \cdot V_A(r) \right) \]

- If SA wave packet cannot live long, why to care about SA interacting with fast ions?
- Radial gradients of plasma may modify such waves of comparable wavelength.
INHOMOGENEOUS PLASMA - 1

- Consider in detail the model: 1-D slab non-uniform cold plasma, \( n_0 = n_0(x), \ P_0 = 0, \ B_0 = B_0 e_z \),

- The presence of plasma gradients modifies our MHD equation. An externally excited electromagnetic wave with perturbed \( \phi \propto \phi(x) \exp(ik_y y - i \omega t) \) is described by

\[
\frac{d}{dx} \left( \omega^2 - \omega_A^2(x) \right) \frac{d\phi}{dx} - k_y^2 \left( \omega^2 - \omega_A^2(x) \right) \phi = 0
\]

\[
\omega_A^2(x) = k_i^2 V_A^2(x)
\]
INHOMOGENEOUS PLASMA - 2

- This equation has zero coefficient at high order derivative at the point \( x = x_0 \) of the local Alfvén resonance layer, where
  \[
  \omega^2 = \omega_A^2(x_0)
  \]

- Investigating the equation in the vicinity of this point:
  \[
  \frac{d}{dx} \left( \omega^2 - \omega_A^2(x) \right) \frac{d\phi}{dx} = 0 \quad \rightarrow \quad \frac{d\phi}{dx} = \frac{\text{const}}{\omega^2 - \omega_A^2(x)}
  \]

- Expand the local Alfvén frequency in the vicinity of the point \( x = x_0 \):
  \[
  \omega^2 = \omega_A^2(x_0) + \frac{d\omega_A^2(x)}{dx}\bigg|_{x=x_0} \cdot (x-x_0)
  \]
  and obtain
  \[
  \phi \propto \text{const} \cdot \ln(x-x_0), \quad x > x_0
  \]
  \[
  \phi \propto \text{const} \cdot \left( \ln|x-x_0| + i\pi \right), \quad x < x_0
  \]

The wave energy peaks up at \( x = x_0 \) and resonant absorption of the wave energy occurs at this point – continuum damping
LINEAR MODE CONVERSION TO KINETIC ALFVÉN WAVE

- Finite ion Larmor radius & finite electron parallel conductivity incorporated in the layer $|x - x_0| \approx \rho_i$ remove the wave singularity
- Wave equation takes the form $(m_e / M_i \ll \beta_e \ll 1)$:

$$\omega^2 \rho_i^2 \nabla^2 (\frac{3}{4} (1 - i\delta_i) + \frac{T_e}{T_i} (1 - i\delta_e)) \nabla_\perp \phi + \frac{d}{dx} (\omega^2 - \omega_A^2 (x)) \frac{d\phi}{dx} - k_y^2 (\omega^2 - \omega_A^2 (x)) \phi = 0$$

- Solution of this equation satisfies the dispersion relation in the form of Kinetic Alfvén Wave:

$$\omega^2 = k_y^2 V_A^2 (x) \left[ 1 + (k_x \rho_i)^2 \left( \frac{3}{4} (1 - i\delta_i) + \frac{T_e}{T_i} (1 - i\delta_e) \right) \right]$$

- In contrast to the shear Alfvén wave, KAW propagates across $B_0$, $\partial \omega_{KAW} / \partial k_x \neq 0$, and it has $E_\parallel \neq 0$
LINEAR MODE CONVERSION TO KINETIC ALFVÉN WAVE

\[ E_x \]

\[ E_0 / Kx \]

\[ KINETIC \ ALFVÉN \ WAVE \]

\[ 0 \]

\[ d \]
DO ANY SHEAR ALFVÉN WAVES EXIST WHICH DO NOT SATISFY A LOCAL SA DISPERSION RELATION AND ARE FREE OF THE CONTINUUM DAMPING?
DISCOVERY OF GLOBAL ALFVÉN EIGENMODE

- In cylindrical geometry, in addition to the continuous Alfvén spectrum, 
  \( \omega^2 = \omega_A^2(r) \equiv k^2(r)V_A^2(r) \), a discrete Global Alfvén Eigenmode with frequency 

The ideal plasma-coil-wall system used in the numerical investigation

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A new high-quality, $Q \equiv \omega/\gamma \sim 10^3$, resonance was discovered during these Alfvén antenna studies, in plasmas with current.

**Real part of the coil impedance vs normalized frequency**
GLOBAL ALFVÉN EIGENMODE

- GAE with $\omega_{GAE} < \omega_A$ exists in Ideal MHD if the current profile determining $dB_\parallel/dr$ provides a minimum in Alfvén continuum

$$\frac{d\omega_A(r)}{dr} \bigg|_{r=r_0} = 0 \quad \text{i.e.} \quad \frac{1}{k_\parallel} \frac{dk_\parallel}{dr} = - \frac{1}{V_A} \frac{dV_A}{dr}$$

- The local minimum of the Alfvén continuum provides a maximum of the perpendicular refraction index $N_r = c k_r / \omega$. Similarly to fiber optics, the electromagnetic wave has to propagate in a “wave-guide” surrounding the region of the extremum refraction index.
The eigenfrequency of GAE does *not* satisfy the local Alfvén resonance condition, \( \omega_{GAE} \neq \omega_A(r) \), for \( 0 < r/a < 1 \). Therefore, this SA mode has *no* singularity and does *not* experience continuum damping.
SUMMARY

- Linearised ideal MHD equations give the equation for plasma displacement

\[ \frac{\partial^2 \xi}{\partial t^2} = c_s^2 \nabla \text{div} \xi + V_A^2 \nabla_\perp \text{div} \xi_\perp + \nabla_\perp \times \mathbf{E} \]

which describes compressional Alfvén, slow magnetosonic, and shear Alfvén waves.

- The compressional waves can be decoupled at low-\( \beta \), and they are coupled otherwise.

- In shear Alfvén wave the fluid displacement vector \( \xi \) and \( \mathbf{E} \) are perpendicular to the magnetic field \( \mathbf{B}_0 \). The wave propagates along \( \mathbf{B}_0 \): \( \omega = \pm k_\parallel V_A \)

- In inhomogeneous plasma, shear Alfvén wave experiences strong continuum damping.

- The continuum damping in ideal MHD corresponds to linear mode conversion to short waved-length Kinetic Alfvén Wave, which has a radial group velocity.

- In cylindrical plasma with current, a discrete eigenmode may exist, Global Alfvén Eigenmode, which has NO continuum damping.