MHD equilibrium and stability in toroidal fusion plasmas

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My/group MHD experience: Equilibrium

- 1D equilibria — Multi region relaxed MHD model: with Hole (2007)
- 2D equilibria — Generalized Taylor states: with Bhattacharjee (c. 1980 when student at PPPL)
- 3D equilibria
  - VMEC studies: Gardner, Blackwell (1992)
  - Improvement of Hayashi (NIFS) HINT code: Gardner & student Lloyd (c. 2000)
  - New Multi-region relaxed MHD code SPEC: with Hole, Hudson (PPPL), student McGann (current work)
Spectrum — Ideal MHD

- Ideal global stable & unstable spectrum
  - 1D with Hole and student Bertram
  - 2D with Grimm, Manickam (PPPL) ➔ PEST2 (1983)
  - 3D WKB-ballooning theory: with student Cuthbert (c. 2000)
  - 3D Eigenmode calculations, through collabs with W.A. Cooper (Lausanne) ➔ Terpsichore;
    C. Nührenberg (Greifswald) ➔ CAS3D
    (current and recent work)
Spectrum — Resistive MHD

- *Resistive* global spectrum
  - 1D Resistive: with Davies (1984) - sparked by Ryu (when student of Grimm at PPPL)
  - 2D Generalized $\Delta'$: with student Pletzer $\rightarrow$ PEST3 (1991)
  - 3D full-mode: with Storer, Gardner & student McMillan $\rightarrow$ SPECTOR3D (c. 2005)
3D Toroidal plasma equilibrium

Good model for toroidal fusion plasma steady state is force balance for total pressure $p$ combined with Ampère’s law relating magnetic field $\mathbf{B}$ and current density $\mathbf{J}$:

$$
\nabla p = \mathbf{J} \times \mathbf{B}, \quad \nabla \times \mathbf{B} = \mathbf{J}, \quad \nabla \cdot \mathbf{B} = 0
$$

EG Stellarators—intrinsically 3D, i.e. no continuous symmetry: and Tokamaks, (also 3D due to coil ripple or instabilities):
Problem with 3D equilibria — they are generically non-integrable

FIG. 7. Poincaré plot of Beltrami field: perturbation of outer boundary $\delta = 0.0030$, with Fourier resolution $M=7, N=2$. Shown in the upper half plot is the coordinates.
Variational method for chaotic field
(Woltjer–)Taylor relaxation principle

Minimize total energy: \[ W = \int_{\text{Plasma}} \frac{B^2}{2} \, d\tau \]

Under constraint of total helicity: \[ K_0 = \frac{1}{2} \int_P (\mathbf{A} \cdot \mathbf{B}) \, d\tau \]

Euler–Lagrange eqns. for \( \delta F = 0 \): \[ F = W - \lambda K_0 \]
give Beltrami equation:

\[ \nabla \times \mathbf{B} = \lambda \mathbf{B} \]

\( \lambda = \text{const} \)

(To check whether \( F \) really is a minimum, need second variation — stability criterion similar to resistive stability, since reconnection is allowed.)
Generalized relaxation principle


Energy principle with global invariants

**Idea:** Extremize total energy

\[ W = \int_{pUV} \left( \frac{B^2}{2} + \frac{p}{\gamma - 1} \right) d\tau \]

subject to *finite* number of ideal-MHD constraints (*unlike* ideal MHD where flux and entropy are “frozen in” to each fluid element — *infinite* no. of constraints).

Require constraints to be a **subset** of the ideal-MHD constraints, so generated states are ideal equilibria:

Spaces of allowed variations:

- **Ideal MHD:** infinity of constraints
- **Relaxed MHD:** finite no. of constraints
- Generalized Taylor equilibria
- Kruskal–Kulsrud equilibria — *include* Taylor states

Generalized relaxation principle

Extremize total energy

Energy principle with global invariants
• ARC Project to find Taylor-relaxed (Beltrami) states separated by KAM* surface

\[ \nabla \times \mathbf{B} = \lambda_i \mathbf{B} \]

*KAM surfaces separate regions of magnetic-field-line chaos and magnetic islands
Numerical equilibrium composed of multiple relaxation regions separated by barriers
Conclusion

• Previous work
  - 3D equilibria (HINT)
  - 3D WKB-ballooning
  - 2D & 3D resistive modes

• Current work
  - 3D MRXMHD equilibria (SPEC)
  - 3D stability theory (CAS3D)